Exercise 1

Let $u \in \mathcal{S}'_h(\mathbb{R}^d)$ and $u_N = \sum_{|j|\le N} \dot\Delta_j u$.

Show that

$$\sum_{j\ge 0} 2^{js} \|\dot\Delta_j\big(\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p} \le C\sup{}_{j\ge 0} 2^{j([s]+2)} \|\dot\Delta_j\big(\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p}$$
$$\le C\|\nabla^{[s]+2}\big(\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p}.$$

Exercise 2

Let $\sigma < 0$ and $(p, r) \in [1, \infty]^2$. Show that

$$\Big\|\big(2^{j\sigma}\|\dot S_j u\|_{L^p}\big)\Big\|_{\ell^r} \le C\|u\|_{\dot B^\sigma_{p,r}}.$$

Exercise 3

Let $s \in [0, \frac{d}{2})$. Show that

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}}\,\mathrm{d}x \le C\|f\|^2_{\dot H^s}.$$

Exercise 4

Let $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$, $v : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a vector field and $\tilde v = \sum_{j\ge 0} \Delta_j v$. Show that

- $$\Big\|\big(2^{js}\|[\Delta_{-1}v \cdot \nabla, \Delta_j]\rho\|_{L^p}\big)_{j\ge -1}\Big\|_{\ell^r} \le C\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{B^{s-1}_{p,r}},$$

- $$\Big\|\big(2^{js}\|[T_{\tilde v^k}, \Delta_j]\partial_k\rho\|_{L^p}\big)_{j\ge -1}\Big\|_{\ell^r} \le C\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{B^{s-1}_{p,r}},$$

- $$\Big\|\big(2^{js}\|\Delta_j \sum_{k=1}^d R(\tilde v^k, \partial_k\rho)\|_{L^p}\big)_{j\ge -1}\Big\|_{\ell^r} \le C\|\nabla v\|_{B^{\frac{d}{p}}_{p,\infty}}\|\nabla\rho\|_{B^{s-1}_{p,r}},$$

  with $s > \max\{-\frac{d}{p}, -\frac{d}{p'}\}$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Exercise 5

- Let E be a Banach space, $\Omega \subset E$ an open set, $I \ni 0$ an open time interval and $X_0 \in \Omega$. Let $v : I \times \Omega \mapsto E$ be $L^1_{\text{loc}}(I; \text{Lip}(\Omega; E))$ in the following sense:

$$\int_K \sup_{\{(X_1(t), X_2(t)) \in \Omega^2 \,|\, X_1(t) \neq X_2(t)\}} \frac{\|v(t, X_1(t)) - v(t, X_2(t))\|_E}{\|X_1(t) - X_2(t)\|_E} dt < \infty, \quad \forall K \text{ compact set in } I.$$

  Show that there exists an open time interval $J \ni 0$ (with $J \subset I$) such that the equation

$$X(t) = X_0 + \int_0^t v(t', X(t'))\, \mathrm{d}t' \,,$$

  i.e. $\frac{d}{dt} X(t) = v(t, X(t))$ with $X(0) = X_0$ has a unique continuous solution $X = X(t) : J \mapsto \Omega$.

- Let $E = C(I; \mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $v \in L^1_{\text{loc}}(I; \text{Lip}(\mathbb{R}^d; \mathbb{R}^d))$. Show that the flow $\psi_t = \psi(t, \cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ of the vector field v:

$$\psi_t(x) = x + \int_0^t v(t', \psi_{t'}(x))\, \mathrm{d}t'$$

  is a C^1 diffeomorphism on $\mathbb{R}^d$ on the entire time interval I and satisfy

$$\|\nabla \psi_t^{\pm 1}\|_{L^\infty} \leq e^{\int_0^t \|\nabla v\|_{L^\infty}},$$
$$\|\nabla \psi_t^{\pm 1} - \text{Id}\,\|_{L^\infty} \leq e^{\int_0^t \|\nabla v\|_{L^\infty}} - 1.$$