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Notes for

Lecture (WS21 0106200): Fourier Analysis and its Applications to PDEs [1]

[1]These lecture notes are not complete (and will be the extension of the lecture notes for the same lecture given by myself in SS19), and are only for the participants of the lecture (WS21 0106200) at the Karlsruhe Institute of Technology. Corrections are welcome to be sent to me by email.

Contents

1 Fourier transform **3**

1.1 Definition on $L^1(\mathbb{R}^d)$ . . . 3

1.2 Schwartz space $\mathcal{S}(\mathbb{R}^d)$ . . . 5

1.2.1 Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ . . . 6

1.2.2 Inverse Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ . . . 7

1.3 Tempered distribution space $\mathcal{S}'(\mathbb{R}^d)$ . . . 9

1.3.1 Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$ . . . 11

1.3.2 Some typical examples . . . 13

1.4 Fourier transform on $L^p(\mathbb{R}^d)$ . . . 15

1.4.1 Some basic facts for $L^p(\mathbb{R}^d)$ revisited . . . 15

1.4.2 Fourier transform on $L^p(\mathbb{R}^d)$, $p \in [1, 2]$ . . . 18

1.4.3 Bernstein-type inequalities . . . 20

1.5 Stationary phase estimates . . . 24

1.5.1 Van der Corput's lemma . . . 24

1.5.2 Higher dimensional case . . . 26

1.6 A dip on Fourier series . . . 28

2 Littlewood-Paley theory **31**

2.1 Homogeneous Besov spaces . . . 32

2.1.1 Homogeneous Besov spaces . . . 33

2.1.2 Sobolev spaces and Lebesgue spaces . . . 38

2.1.3 Homogeneous paradifferential calculus . . . 40

2.2 Nonhomogeneous Besov spaces . . . 44

2.2.1 Nonhomogeneous Besov spaces . . . 44

2.2.2 Commutator estimates . . . 45

3 Applications to PDEs **50**

3.1 Transport equation . . . 50

3.1.1 Flow of a Lipschitz-continuous vector field . . . 53

3.1.2 The L^p-framework . . . 56

3.1.3 The $B^s_{p,r}$-framework . . . 63

3.2 Navier-Stokes equations . . . 67

3.2.1 Weak solutions . . . 68

3.2.2 Strong solutions . . . 70

Reference:
H. Bahouri, J.-Y. Chemin and R. Danchin: Fourier analysis and nonlinear partial differential equations. Springer, 2011.

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1 Fourier transform

In this section we will introduce the Fourier transform in the whole space setting $\mathbb{R}^d$, $d \geqslant 1$.

1.1 Definition on $L^1(\mathbb{R}^d)$

We endow always the whole space $\mathbb{R}^d$, $d \geqslant 1$ with the Lebesgue measure m^d, which is the complete measure defined on the so-called Lebesgue σ-algebra (as the completion of the Borel σ-algebra). In particular, the Lebesgue measure of a box

$$\Pi_{j=1}^d [a_j, b_j], \quad -\infty < a_j \leqslant b_j < \infty,$$

in $\mathbb{R}^d$ is $\Pi_{j=1}^d (b_j - a_j)$.
A real-valued function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is called Lebesgue measurable if for any $\alpha \in \mathbb{R}$ the set

$$\{x \in \mathbb{R}^d \mid f(x) > \alpha\}$$

is Lebesgue measurable. A complex-valued function $f : \mathbb{R}^d \mapsto \mathbb{C}$ is Lebesgue measurable if its real and imaginary parts are both Lebesgue measurable.
We consider the integrable function on $\mathbb{R}^d$:

$$f \in L^1(\mathbb{R}^d; \mathbb{C}) = \Big\{ f \text{ is measurable on } \mathbb{R}^d \mid \|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)| \,\mathrm{d}x < \infty \Big\} / \sim,$$

where the Lebesgue integral is defined as the Riemann integral:

$$\int_{\mathbb{R}^d} |f(x)| \,\mathrm{d}x = \int_0^\infty m^d(\{x \in \mathbb{R}^d \mid |f(x)| > \alpha\}) d\alpha,$$

and we call two measurable functions equivalent $f \sim g$ if $m^d(\{f \neq g\}) = 0$. Similarly, a measurable function $f : \mathbb{R}^d \mapsto \mathbb{C}$ is called p-integrable, $p \in [1, \infty)$ if $|f|^p$ is integrable and we denote its $L^p(\mathbb{R}^d)$-norm as follows

$$\|f\|_{L^p(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |f(x)|^p \,\mathrm{d}x \Big)^{1/p} = \Big(p \int_0^\infty \alpha^{p-1} m^d(\{x \in \mathbb{R}^d \mid |f(x)| > \alpha\}) d\alpha \Big)^{\frac{1}{p}}.$$

If $p = \infty$, then we say that the function f is essentially bounded, if it has finite L^∞-norm:

$$\|f\|_{L^\infty(\mathbb{R}^d)} = \inf_\alpha \{m^d(\{x \in \mathbb{R}^d \mid |f(x)| > \alpha\}) = 0\}.$$

We define $L^p(\mathbb{R}^d)$ as the space of equivalence classes of p-integrable functions, which is Banach space.

Definition 1.1. *Let $f \in L^1(\mathbb{R}^d;\mathbb{C})$, $d \geqslant 1$. We define its Fourier transform as a function $\hat{f} \in L^\infty(\mathbb{R}^d;\mathbb{C})$ below*

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x\,, \quad \forall \xi \in \mathbb{R}^d\,.$$

Proposition 1.1 (Riemann-Lebesgue)**.** *Let $f \in L^1(\mathbb{R}^d;\mathbb{C})$, then its Fourier transform $\hat{f}$ is continuous and satisfies*

$$\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0.$$

Proof. **Exercise.** We make use of the continuity of the function in ξ: $e^{-ix\cdot\xi}$ and the Lebesgue convergence theorem to show the continuity of $\hat{f}(\xi)$. The decay at infinity of $\hat{f}(\xi)$ follows from the density argument and integration by parts. □

By Proposition 1.1, the Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}^d) \mapsto C_b(\mathbb{R}^d) \supset C_0(\mathbb{R}^d)$$

is a linear continuous map and

$$\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^d)} \leqslant \frac{1}{(2\pi)^{\frac{d}{2}}} \|f\|_{L^1(\mathbb{R}^d)}, \text{ i.e. } \|\mathcal{F}\|_{L^1(\mathbb{R}^d)\mapsto L^\infty(\mathbb{R}^d)} \leqslant \frac{1}{(2\pi)^{\frac{d}{2}}}.$$

In the above, $C_0(\mathbb{R}^d)$ is the space of continuous functions which decay to 0 at infinity, and $C_b(\mathbb{R}^d)$ is the space of bounded continuous functions endowed with the supremum norm.

If $f, g \in L^1(\mathbb{R}^d;\mathbb{C})$, then

$$\begin{aligned}
&\text{The translated function } \tau_{x_0}(f), \text{ with } \big(\tau_{x_0}(f)\big)(x) = f(x - x_0), \quad x_0 \in \mathbb{R}^d,\\
&\text{The modulated function } e^{ix\cdot\xi_0} f(x), \text{ with } \xi_0 \in \mathbb{R}^d,\\
&\text{The rescaled function } f \circ A, \text{ with } (f \circ A)(x) = f(Ax), \ A \in \mathbb{R}^{d\times d} \text{ an invertible matrix},\\
&\text{The convolution } f * g, \text{ with } (f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y\,,
\end{aligned}$$

are all in $L^1(\mathbb{R}^d;\mathbb{C})$. We investigate their Fourier transform.

Lemma 1.1. *Let $x_0, \xi_0 \in \mathbb{R}^d$, $f, g \in L^1(\mathbb{R}^d; \mathbb{C})$ and A be a real invertible $d \times d$ matrix. Then*

$$\mathcal{F}(\tau_{x_0}(f)) = e^{-ix_0\cdot\xi}\mathcal{F}(f),$$
$$\mathcal{F}(e^{ix\cdot\xi_0}f) = \tau_{\xi_0}\mathcal{F}(f),$$
$$\mathcal{F}(f \circ A) = |\det A|^{-1}\mathcal{F}(f) \circ A^{-T}, \text{ and in particular } \mathcal{F}(f(\lambda\cdot)) = \lambda^{-d}(\mathcal{F}(f))(\lambda^{-1}\cdot),\ \forall\lambda > 0,$$
$$\mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}}\mathcal{F}(f)\mathcal{F}(g),$$
$$\int_{\mathbb{R}^d} f\mathcal{F}(g)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{F}(f)g\,\mathrm{d}x\,.$$

Proof. **Exercise.** □

1.2 Schwartz space $\mathcal{S}(\mathbb{R}^d)$

In this subsection we study the Fourier transform of Schwartz functions, i.e. smooth rapidly decaying functions.

Definition 1.2 (Schwartz space)**.** *The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the set of the smooth functions $f \in C^\infty(\mathbb{R}^d)$ satisfying for any $k \in \mathbb{N}$*

$$\|f\|_{k,\mathcal{S}} := \sup_{x\in\mathbb{R}^d,\,|\alpha|\leqslant k}(1 + |x|^k)|\partial^\alpha f(x)| < \infty.$$

Remark 1.1. *It is equivalent to say that*

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d)\,|\,\sup_{x\in\mathbb{R}^d}|x^\alpha\partial^\beta f(x)| < \infty,\ \forall \text{ multiindices } \alpha, \beta\}.$$

Notice that for $f \in \mathcal{S}$, $\|f\|_{k,\mathcal{S}}$ may depend on $k \in \mathbb{N}$.
It is also easy to see that

- *$\mathcal{S}(\mathbb{R}^d) \subset C^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$ and $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$;*
- *If $f \in \mathcal{S}(\mathbb{R}^d)$, then $x^\alpha f$, $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^d)$ for any multiindex α;*
- *If $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in C_b^\infty(\mathbb{R}^d)$, then $fg \in \mathcal{S}(\mathbb{R}^d)$;*
- *If $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in \mathcal{D}'(\mathbb{R}^d) = (C_0^\infty(\mathbb{R}^d))'$ with compact support, then $f * g \in \mathcal{S}(\mathbb{R}^d)$;*
- *If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $f * g \in \mathcal{S}(\mathbb{R}^d)$;*
- *The Gaussian function $e^{-\frac{1}{2}|x|^2} \in \mathcal{S}(\mathbb{R}^d)$.*

We say that $f_n \to f$ in $\mathcal{S}$ if $\|f_n - f\|_{k,\mathcal{S}} \to 0$ for all $k \in \mathbb{N}$. We can then introduce a metric $d(\cdot,\cdot)$ on $\mathcal{S}$:

$$d(f,g) := \sum_{k\in\mathbb{N}} 2^{-k} \frac{\|f-g\|_{k,\mathcal{S}}}{1+\|f-g\|_{k,\mathcal{S}}},$$

such that $d(f_n, f) \to 0$ if and only if $f_n \to f$ in $\mathcal{S}$.

Proposition 1.2. *The space $(\mathcal{S}(\mathbb{R}^d), d(\cdot,\cdot))$ is a complete metric space and the space $\mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$ of smooth compactly supported functions is dense in it.*

Proof. **Exercise.** Observe that for the smooth cutoff function χ, $\|f - \chi(R^{-1}\cdot)f\|_{k,\mathcal{S}} \leqslant C_k R^{-1} \|f\|_{k+1,\mathcal{S}} \to 0$ as $R \to \infty$. □

1.2.1 Fourier transform on $\mathcal{S}(\mathbb{R}^d)$

Since $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, we can define the Fourier transform on $\mathcal{S}(\mathbb{R}^d)$:

Theorem 1.1. *The Fourier transform maps continuously from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$: For any integer $k \in \mathbb{N}$, there exist a constant C and an integer $N \in \mathbb{N}$ such that*

$$\|\mathcal{F}(f)\|_{k,\mathcal{S}} \leqslant C\|f\|_{N,\mathcal{S}}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Furthermore, the equalities in Lemma 1.1 together with the following equalities hold true:

$$\mathcal{F}((\frac{1}{i}\partial_x)^\alpha f) = \xi^\alpha \mathcal{F}(f), \quad \mathcal{F}(x^\alpha f) = (i\partial_\xi)^\alpha \mathcal{F}(f), \quad \forall \textit{multiindex } \alpha. \tag{1.1}$$

Proof. It is straightforward to use Lebesgue's theorem and integration by parts to show the above two equalities. Then for any $|\alpha| + |\beta| \leqslant k$, there exist C and N (we can simply take $N = k + d + 1$) such that

$$\begin{aligned}\left|\xi^\alpha \partial_\xi^\beta \mathcal{F}(f)\right| &= \left|\mathcal{F}\big((\frac{1}{i}\partial_x)^\alpha (\frac{1}{i}x)^\beta f\big)\right| \leqslant (2\pi)^{-\frac{d}{2}} \|\partial_x^\alpha (x^\beta f)\|_{L^1} \\ &\leqslant \|(1+|x|)^{-d-1}\|_{L^1} \|(1+|x|)^{d+1}\partial_x^\alpha(x^\beta f)\|_{L^\infty} \leqslant C\|f\|_{N,\mathcal{S}}.\end{aligned}$$

□

Corollary 1.1. *The Fourier transform maps the Gaussian function $e^{-\frac{1}{2}|x|^2} \in \mathcal{S}(\mathbb{R}^d)$ to itself: $\mathcal{F}(e^{-\frac{1}{2}|x|^2}) = e^{-\frac{1}{2}|\xi|^2}$.*

Proof. We first notice that

$$(\partial_{x_j} + x_j)e^{-\frac{1}{2}|x|^2} = 0, \quad j = 1, \cdots, d,$$

and hence by Theorem 1.2,

$$i(\xi_j + \partial_{\xi_j})\mathcal{F}(e^{-\frac{1}{2}|x|^2}) = 0, \quad j = 1, \cdots, d.$$

Let $d = 1$, then the function $\phi(\xi) := \mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi) \in \mathcal{S}(\mathbb{R})$ satisfies the following first-order differential equation

$$\phi' + \xi\phi = 0, \text{ i.e. } (e^{\frac{1}{2}\xi^2}\phi)' = 0,$$

and hence there exists a constant $C \in \mathbb{R}$ such that

$$\phi(\xi) = Ce^{-\frac{1}{2}\xi^2}.$$

In particular,

$$C = \phi(0) = \mathcal{F}(e^{-\frac{1}{2}x^2})(0) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\,\mathrm{d}x = 1,$$

and thus

$$\mathcal{F}(e^{-\frac{1}{2}x^2}) = \phi(\xi) = e^{-\frac{1}{2}\xi^2}.$$

For $d \geqslant 2$, we simply notice that

$$\begin{aligned}
\mathcal{F}(e^{-\frac{1}{2}|x|^2}) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}e^{-\frac{1}{2}|x|^2}\,\mathrm{d}x \\
&= \frac{1}{(2\pi)^{\frac{d-1}{2}}}\int_{\mathbb{R}^{d-1}} e^{-ix'\cdot\xi'}e^{-\frac{1}{2}|x'|^2}\,\mathrm{d}x'\,\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{-ix_d\cdot\xi_d}e^{-\frac{1}{2}(x_d)^2}\,\mathrm{d}x_d \\
&= \frac{1}{(2\pi)^{\frac{d-1}{2}}}\int_{\mathbb{R}^{d-1}} e^{-ix'\cdot\xi'}e^{-\frac{1}{2}|x'|^2}\,\mathrm{d}x'\,e^{-\frac{1}{2}(\xi_d)^2} = \mathcal{F}(e^{-\frac{1}{2}|x'|^2})e^{-\frac{1}{2}(\xi_d)^2},
\end{aligned}$$

with $x' = (x_1, \cdots, x_{d-1})$, $\xi' = (\xi_1, \cdots, \xi_{d-1})$. An easy induction argument implies the result for $d \geqslant 2$. □

1.2.2 Inverse Fourier transform on $\mathcal{S}(\mathbb{R}^d)$

Definition 1.3. *Let $f \in L^1(\mathbb{R}^d)$. We define*

$$\check{f}(x) := \mathcal{F}^{-1}(f)(x) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{ix\cdot\xi}f(\xi)\,\mathrm{d}\xi\,.$$

Theorem 1.2. *Let $f \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x) = \mathcal{F}(f(-\cdot))(x),$$
$$\text{that is, } \mathcal{F}^{-1} = \mathcal{F} \circ R = R \circ \mathcal{F},$$

with R denoting the flip operator $(Rf)(x) = f(-x)$ on $\mathcal{S}(\mathbb{R}^d)$, and

$$\mathcal{F}^{-1}\mathcal{F}(f)(x) = f(x) = \mathcal{F}\mathcal{F}^{-1}(f)(x),$$
$$\text{that is, } \mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathrm{Id} \ \text{ and } \mathcal{F}^2 = R \text{ on } \mathcal{S}(\mathbb{R}^d).$$

The Fourier transform is an automorphism on $\mathcal{S}(\mathbb{R}^d)$.

Proof. As $\lim_{\varepsilon\to 0} e^{-\frac{\varepsilon^2}{2}|x|^2} = 1$ pointwisely, we calculate straightforward

$$\begin{aligned}
\mathcal{F}^{-1}\mathcal{F}(f)(x) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat{f}(\xi)\,\mathrm{d}\xi = \lim_{\varepsilon\to 0}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{ix\cdot\xi} \hat{f}(\xi)\,\mathrm{d}\xi \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{ix\cdot\xi} \int_{\mathbb{R}^d} e^{-iy\cdot\xi} f(y)\,\mathrm{d}y\ \mathrm{d}\xi \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{\varepsilon^2}{2}|\xi|^2} e^{-i(y-x)\cdot\xi}\,\mathrm{d}\xi\, f(y)\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathcal{F}(e^{-\frac{\varepsilon^2}{2}|\xi|^2})(y-x) f(y)\,\mathrm{d}y\,.
\end{aligned}$$

We recall Lemma 1.1 and Corollary 1.1 to derive that

$$(2\pi)^{-\frac{d}{2}}\mathcal{F}(e^{-\frac{\varepsilon^2}{2}|\xi|^2})(z) = (2\pi)^{-\frac{d}{2}}\varepsilon^{-d} e^{-\frac{1}{2\varepsilon^2}|z|^2},$$

which is a Dirac sequence. Hence $\mathcal{F}^{-1}\mathcal{F}(f)(x) = f(x)$ and $\mathcal{F}\mathcal{F}^{-1} = \mathrm{Id}$ follows simply from $\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x)$. □

[22.10.2021]

[27.10.2021]

Remark 1.2. *If $f \in \mathcal{S}(\mathbb{R}^d)$, then*

$$e^{-\frac{\varepsilon^2}{2}|x|^2} f \to f \ \text{ in } \mathcal{S}(\mathbb{R}^d) \ \text{ as } \varepsilon \to 0.$$

Hence

$$(2\pi)^{-\frac{d}{2}}\varepsilon^{-d} e^{-\frac{1}{2\varepsilon^2}|x|^2} * f \to f \ \text{ as } \mathcal{S}(\mathbb{R}^d) \ \text{ as } \varepsilon \to 0.$$

Corollary 1.2. *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}} \mathcal{F}(f)\mathcal{F}(g), \quad \mathcal{F}(fg) = (2\pi)^{-\frac{d}{2}} \mathcal{F}(f) * \mathcal{F}(g),$$

and

$$\int_{\mathbb{R}^d} f\bar{g}\, \mathrm{d}x = \int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}}\, \mathrm{d}\xi\,.$$

In particular, the Parseval's identity holds: $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$.

Proof. We show the result by the following equalities from Lemma 1.1:

$$\mathcal{F}(f * g) = (2\pi)^{\frac{d}{2}} \mathcal{F}(f)\mathcal{F}(g), \quad \int_{\mathbb{R}^d} f\hat{g} = \int_{\mathbb{R}^d} \hat{f}g,$$

and the equalities in Theorem 1.2. □

1.3 Tempered distribution space $\mathcal{S}'(\mathbb{R}^d)$

We recall first the definitions of the distribution space $\mathcal{D}'(U)$ on an open set U. Let $U \subset \mathbb{R}^d$ be open, and

$$\mathcal{D}(U) = C_0^\infty(U) = C_c^\infty(U)$$

be the vector space of infinitely differential functions with compact support (i.e. the so-called test functions). We say

$$f_j \to f \text{ in } \mathcal{D}(U) = C_0^\infty(U),$$

if there exists a compact set $K \subset U$ such that

$$\mathrm{Supp}\,(f_j) \subset K \quad \forall j, \text{ and } \partial^\alpha f_j \to \partial^\alpha f \text{ in } C_b(U), \quad \forall \alpha \in \mathbb{N}^d\,.$$

A distribution $T \in \mathcal{D}'(U)$ is a continuous linear map from $C_0^\infty(U)$ to $\mathbb{C}$: For any compact set K in U, there exists $k \in \mathbb{N}$ and $C > 0$ such that

$$|T(f)| \leqslant C\|f\|_{C_b^k(K)}, \quad \forall f \in \mathcal{D}(U) \text{ with } \mathrm{Supp}\,(f) \subset K.$$

We say

$$T_n \to T \text{ in } \mathcal{D}'(U),$$

if

$$T_n(f) \to T(f), \quad \forall f \in \mathcal{D}(U).$$

Then for any compact set $K \subset U$, there exists k and C such that

$$|T_n(f)| \leqslant C\|f\|_{C_b^k(K)}, \quad \forall f \in \mathcal{D}(U) \text{ with } \mathrm{Supp}\,(f) \subset K,$$

and

$$\sup_{f\in\mathcal{D}(U) \text{ with } \mathrm{Supp}\,(f)\subset K, \|f\|_{C_b^k(K)}\leqslant 1} |T_n(f) - T(f)| \to 0.$$

Let $T \in \mathcal{D}'(\mathbb{R}^d)$ be a distribution. We define the support of T: $\mathrm{Supp}\,(T)$ as the complement of the following set

$$\{x \in \mathbb{R}^d \mid \exists r > 0 \text{ s.t. } T(f) = 0, \quad \forall f \in C_0^\infty(B(x,r))\}.$$

Definition 1.4 (Tempered distribution)**.** *A tempered distribution on $\mathbb{R}^d$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathbb{C}$. We denote the set of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We say that $T_j \to T$ in $\mathcal{S}'(\mathbb{R}^d)$ if*

$$T_j(f) \to T(f) \quad \textit{for any } f \in \mathcal{S}(\mathbb{R}^d).$$

Remark 1.3.

- *By the definition above, a tempered distribution T is a distribution $T \in \mathcal{D}'(\mathbb{R}^d) = (C_0^\infty(\mathbb{R}^d))'$ such that there exists $k \in \mathbb{N}$ and $C \in \mathbb{R}$ s.t.*

  $$|T(f)| \leqslant C\|f\|_{k,\mathcal{S}}, \quad \forall f \in \mathcal{D}(\mathbb{R}^d).$$

  Indeed, if $T \in \mathcal{D}'(\mathbb{R}^d)$ such that the above inequality holds, then as $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ is dense, there exists a unique continuation of T as a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$.

- *The convergence $T_j \to T$ in $\mathcal{S}'(\mathbb{R}^d)$ means indeed that there exists $k \in \mathbb{N}$ such that*

  $$\sup_{\|f\|_{k,\mathcal{S}}\leqslant 1} |T_j(f) - T(f)| \to 0$$

Example 1.1.

- *Every function $g \in L^1(\mathbb{R}^d)$ defines a tempered distribution $T_g \in \mathcal{S}'(\mathbb{R}^d)$ as*

  $$T_g(f) = \int_{\mathbb{R}^d} gf \,\mathrm{d}x\,, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

  *Similarly it holds for $g \in L^p(\mathbb{R}^d)$ for any $p \in [1,\infty]$. (**Exercise**)*

- *Let us denote the set of distributions with compact support by $\mathcal{E}'(\mathbb{R}^d)$, which is the dual space of $C^\infty(\mathbb{R}^d)$ endowed with the seminorms*

  $$\sup_{|\alpha|\leqslant k} \|\partial^\alpha f\|_{L^\infty(B(0,k))}, \quad k \in \mathbb{N}\,.$$

  Then $\mathcal{E}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$ and in particular the Dirac function $\delta_0 \in \mathcal{E}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$.

1.3.1 Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$

If $A : \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{S}(\mathbb{R}^d)$ is a linear operator, then by duality we can define the operator $A^t : \mathcal{S}'(\mathbb{R}^d) \mapsto \mathcal{S}'(\mathbb{R}^d)$ as follows:

$$(A^t T)(f) = T(Af), \quad \forall T \in \mathcal{S}'(\mathbb{R}^d), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Definition 1.5. *Let $A \in \mathbb{R}^{d\times d}$ an invertible matrix, $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$. Then we define the following operators on $\mathcal{S}'(\mathbb{R}^d)$:*

- *The translation $(\tau_{x_0} T)(f) = T(\tau_{-x_0} f)$,*
- *The modulation $(e^{ix\cdot\xi_0} T)(f) = T(e^{ix\cdot\xi_0} f)$,*
- *The rescaling $(T \circ A)(f) = |\det(A)|^{-1} T(f \circ A^{-1})$,*
- *The multiplication by $x_j : (x_j T)(f) = T(x_j f)$,*
- *The multiplication by $g : (gT)(f) = T(fg)$,*
- *The derivative $\partial_{x_j} : (\partial_{x_j} T)(f) = -T(\partial_{x_j} f)$,*
- *The convolution with $f : (T * f)(x) = T(f(x-\cdot))$ (is a smooth function),*
- *The Fourier transform $\mathcal{F} : \mathcal{F}(T)(f) = T(\mathcal{F}(f))$,*
- *The inverse Fourier transform $\mathcal{F}^{-1}(T)(f) = T(\mathcal{F}^{-1}(f)) = T(\mathcal{F}R(f))$.*

Remark 1.4. *(**Exercise**) We can simply check the above equalities directly when $T = T_g$, $g \in \mathcal{S}(\mathbb{R}^d)$.*

[27.10.2021]

[03.11.2021]

By Lemma 1.1, (1.1), Theorem 1.2, Corollary 1.2 and the above definition, it follows that (**Exercise**)

Theorem 1.3. *The Fourier transform is an automorphism on $\mathcal{S}'(\mathbb{R}^d)$ with*

$$\mathcal{F}^{-1}\mathcal{F} = \mathrm{Id} = \mathcal{F}\mathcal{F}^{-1}, \quad \mathcal{F}^{-1} = \mathcal{F} \circ R = R \circ \mathcal{F}, \quad \mathcal{F}^2 = R, \quad \text{on } \mathcal{S}'(\mathbb{R}^d),$$

and the following rules are valid on $\mathcal{S}'(\mathbb{R}^d)$:

$$\mathcal{F}(\tau_{x_0}(T)) = e^{-ix_0\cdot\xi}\mathcal{F}(T),\ \mathcal{F}(e^{ix\cdot\xi_0}T) = \tau_{\xi_0}\mathcal{F}(T), \quad \forall x_0, \xi_0 \in \mathbb{R}^d,$$
$$\mathcal{F}(T \circ A) = |\det A|^{-1}\mathcal{F}(T) \circ A^{-T},\ \textit{e.g.}\ \mathcal{F}(T(\lambda\cdot)) = \lambda^{-d}(\mathcal{F}(T))(\lambda^{-1}\cdot),\ \forall \lambda > 0,$$
$$\mathcal{F}(T * f) = (2\pi)^{\frac{d}{2}}\mathcal{F}(T)\mathcal{F}(f),\ \mathcal{F}(Tf) = (2\pi)^{-\frac{d}{2}}\mathcal{F}(T) * \mathcal{F}(f), \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$
$$\mathcal{F}((\frac{1}{i}\partial_x)^\alpha T) = \xi^\alpha\mathcal{F}(T),\ \mathcal{F}(x^\alpha T) = (i\partial_\xi)^\alpha\mathcal{F}(T), \quad \forall \textit{multiindex } \alpha.$$

Proposition 1.3. *$\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{S}'(\mathbb{R}^d)$.*

Proof. By Remark 1.2,

$$(2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2} * (e^{-\frac{\delta^2}{2}|x|^2}f) \to f \text{ in } \mathcal{S}(\mathbb{R}^d), \text{ as } \varepsilon \to 0, \quad \delta \to 0.$$

By the definitions, for any $T \in \mathcal{S}'(\mathbb{R}^d)$, there exist k and C such that

$$\begin{aligned}\big|\big(T * g\big)(x)\big| = \big|T(g(x-\cdot))\big| &\leqslant C\|g(x-\cdot)\|_{k,\mathcal{S}} \\ &= C\sup{}_{y\in\mathbb{R}^d,\,|\alpha|\leqslant k}(1+|y|^k)|\partial_y^\alpha g(x-y)|, \quad \forall g \in \mathcal{S}(\mathbb{R}^d),\end{aligned}$$

which is a smooth function whose derivatives of any order grow at most polynomially, and for any $f, g \in \mathcal{S}$,

$$(T * g)(f) = \int_{\mathbb{R}^d} T(g(x-\cdot))\, f(x)\,\mathrm{d}x \ = T\Big(\int_{\mathbb{R}^d} g(x-\cdot)f(x)\,\mathrm{d}x\Big) = T((g\circ R) * f),$$

where $(g\circ R)(x) = g(-x)$. Hence for any $T \in \mathcal{S}'$,

$$\begin{aligned}&T\Big((2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2} * (e^{-\frac{\delta^2}{2}|x|^2}f)\Big) \\ &= \Big(T * (2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2}\Big)(e^{-\frac{\delta^2}{2}|x|^2}f) \\ &= \Big(e^{-\frac{\delta^2}{2}|x|^2}\big((2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2} * T\big)\Big)(f) \to T(f), \text{ as } \varepsilon \to 0, \quad \delta \to 0.\end{aligned}$$

Thus

$$\mathcal{S} \ni e^{-\frac{\delta^2}{2}|x|^2}\big((2\pi)^{-\frac{d}{2}}\varepsilon^{-d}e^{-\frac{1}{2\varepsilon^2}|\cdot|^2} * T\big) \to T \text{ in } \mathcal{S}'.$$

□

Proposition 1.4. *Let $T \in \mathcal{S}'(\mathbb{R}^d)$. Then* $\mathrm{Supp}\,(T) = \{0\} \Leftrightarrow \mathcal{F}(T)$ *is a polynomial.*

Proof. "$\Leftarrow$" is straightforward by considering high-order derivative of $\mathcal{F}(T)$ (**Exercise**).
Now let $\mathrm{Supp}\,(T) = \{0\}$. For any $\varepsilon \in (0,1)$, we take a cutoff function $\chi_\varepsilon = \chi(\varepsilon^{-1}\cdot) \in C_0^\infty(B(0,\varepsilon))$. Then $T((1-\chi_\varepsilon)f) = 0$ for any $f \in \mathcal{S}(\mathbb{R}^d)$ and hence $T(f) = T(\chi_\varepsilon f)$. As $T \in \mathcal{S}'(\mathbb{R}^d)$, there exist k, C such that

$$|T(f)| = |T(\chi_\varepsilon f)| \leqslant C\|\chi_\varepsilon f\|_{k,\mathcal{S}}.$$

We claim that for any $f \in \mathcal{S}(\mathbb{R}^d)$ such that $\partial^\alpha f(0) = 0$ for any $|\alpha| \leqslant k$, $T(f) = 0$. Indeed, for any $|x| \leqslant \varepsilon < 1$, by Taylor's expansion formula,

$$|f(x)| \leqslant C_0\sup{}_{|\beta|=k+1}\|\partial^\beta f\|_{L^\infty}|x|^{k+1},$$

$$|\partial^\gamma f(x)| \leqslant C_0 \sup_{|\beta|=k+1} \|\partial^\beta f\|_{L^\infty} |x|^{k+1-|\gamma|}, \quad \forall |\gamma| \leqslant k.$$

Hence

$$|T(f)| \leqslant C\|\chi(\varepsilon^{-1}\cdot) f\|_{k,\mathcal{S}} \leqslant C_1 \varepsilon \|f\|_{k+1,\mathcal{S}},$$

which tends to 0 as $\varepsilon \to 0$. Thus $T(f) = 0$.
Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then by $\mathrm{Supp}\,(T) = \{0\}$ and Taylor's formula again,

$$T(f) = T\big(f - \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \partial^\alpha f(0) x^\alpha\big) + \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \partial^\alpha f(0) T(\chi x^\alpha),$$

where the first summand on the righthand side vanishes by the above claim.
Let $g = \mathcal{F}^{-1} f$, then

$$\begin{aligned}\hat{T}(g) = T(\hat{g}) = \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} (\partial^\alpha \hat{g})(0) T(\chi x^\alpha) &= \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \big(T(\chi x^\alpha)\big) \int_{\mathbb{R}^d} (-ix)^\alpha g(x)\,\mathrm{d}x \\ &= \Big(\sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \big(T(\chi x^\alpha)\big)(-ix)^\alpha\Big)(g).\end{aligned}$$

□

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[05.11.2021]

1.3.2 Some typical examples

Example 1.2. *Calculate the Fourier transform of the delta function*

$$\mathcal{F}\delta = (2\pi)^{-\frac{d}{2}},$$

where $\delta(f) = f(0)$, *and* $(2\pi)^{-\frac{d}{2}}$ *is a constant function which applies on a Schwartz function as a tempered distribution as follows*

$$T_{(2\pi)^{-\frac{d}{2}}}(f) = \int_{\mathbb{R}^d} (2\pi)^{-\frac{d}{2}} f(x)\,\mathrm{d}x\,, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Thus we have

$$\widehat{\partial^\alpha \delta} = (2\pi)^{-\frac{d}{2}} (i\xi)^\alpha.$$

And hence by Proposition 1.4, the tempered distribution T with $\mathrm{Supp}\,(T) = \{0\}$ *can only be a linear combination of δ and its derivatives:* $T = \sum_{|\alpha| \leqslant k} a_\alpha \partial^\alpha \delta$.

The inverse Fourier transform of the delta function reads also as

$$\mathcal{F}^{-1}(\delta) = (2\pi)^{-\frac{d}{2}},$$

which gives the (inverse) Fourier transform of the constant function 1 *as*

$$\mathcal{F}(1) = \mathcal{F}^{-1}(1) = (2\pi)^{\frac{d}{2}}\delta.$$

Indeed, it is easy to calculate

$$\hat{\delta}(f) = \delta(\hat{f}) = \hat{f}(0) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(\xi)\,\mathrm{d}\xi\ = T_{(2\pi)^{-\frac{d}{2}}}(f).$$

Similarly, we have

$$\mathcal{F}^{-1}(\delta)(f) = \delta(\mathcal{F}^{-1}(f)) = \delta(\mathcal{F}(f)(-\cdot)) = \hat{f}(0) = T_{(2\pi)^{-\frac{d}{2}}}(f).$$

Example 1.3. *When $d = 1$, then $\mathcal{F}(e^{-|x|}) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\xi^2}$ (**Exercise**).*

Example 1.4. *When $\sigma \in (0, d)$, then $\mathcal{F}(|x|^{-\sigma}) = c|\xi|^{\sigma-d}$ for some constant c. (**Exercise**)*

We give the ideas here. Let $d \geqslant 2$. Define the operators

$$S = \sum_{j=1}^{d} x_j \partial_{x_j}, \quad Z_{j,k} = x_j\partial_{x_k} - x_k\partial_{x_j},$$

such that

$$|x|^2\partial_{x_k} = x_k S + \sum_{j=1}^{d} x_j Z_{j,k},$$

and

$$S(|x|^{-\sigma}) = -\sigma|x|^{-\sigma}, \quad Z_{j,k}(|x|^{-\sigma}) = 0.$$

Then we show that

$$\begin{aligned}&S(\mathcal{F}(|x|^{-\sigma})) = -\mathcal{F}(R(|x|^{-\sigma})) - d\mathcal{F}(|x|^{-\sigma}) = (\sigma - d)\mathcal{F}(|x|^{-\sigma}),\\ &Z_{j,k}(\mathcal{F}(|x|^{-\sigma})) = 0,\end{aligned}$$

such that

$$|\xi|^2\partial_{\xi_k}(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma})) = \xi_k S(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma})) + \sum_{j=1}^{d} \xi_j Z_{j,k}(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma})) = 0.$$

Therefore the distribution $\nabla(|\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma}))$ is supported at $\{0\}$ such that $T := |\xi|^{d-\sigma}\mathcal{F}(|x|^{-\sigma}) - c$ is supported at $\{0\}$ for some constant c. Thus by Example 1.2, $T = \sum_{|\alpha|\leqslant k} a_\alpha \partial^\alpha \delta$ and

$$0 = ST = \sum_{|\alpha|\leqslant k} a_\alpha S\big(\partial^\alpha \delta\big) = -\sum_{|\alpha|\leqslant k} a_\alpha (d + |\alpha|)\partial^\alpha \delta,$$

where we used $S(\partial^\alpha\delta) = -(d+|\alpha|)\partial^\alpha\delta$ (from the fact $\partial^\alpha\delta = (\widehat{\frac{1}{i}x)^\alpha(2\pi})^{-\frac{d}{2}})$.
Hence $a_\alpha = 0$, $T = 0$ and $\mathcal{F}(|x|^{-\sigma}) = c|\xi|^{\sigma-d}$.
For $d = 1$, we simply make use of the operator $S = x\frac{d}{dx}$.

Remark 1.5. *If f is homogeneous of degree m: $f(\lambda x) = \lambda^m f(x)$, then $\hat{f}$ is homogeneous of degree $-(m+d)$: $\hat{f}(\lambda^{-1}\xi) = \lambda^{m+d}\hat{f}(\xi)$. This coincides with Example 1.4.*

1.4 Fourier transform on $L^p(\mathbb{R}^d)$

Since (see Example 1.1) any function $f \in L^p(\mathbb{R}^d)$, $p \in [1,\infty]$ can be identified with a tempered distribution $T_f \in \mathcal{S}'(\mathbb{R}^d)$ which is defined as

$$T_f(\phi) = \int_{\mathbb{R}^d} f\phi \,\mathrm{d}x\,, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d),$$

we can define the Fourier transform of f as the following tempered distribution

$$\mathcal{F}(f)(\phi) = T_f(\mathcal{F}(\phi)), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

1.4.1 Some basic facts for $L^p(\mathbb{R}^d)$ revisited

We recall here some basic facts concerning the Lebesgue spaces $L^p(\mathbb{R}^d)$, $p \in [1,\infty]$.

- The Lebesgue space $L^p(\mathbb{R}^d)$, $p \in [1,\infty]$ is Banach space. In particular, $L^2(\mathbb{R}^d)$ is Hilbert space with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x)\bar{g}(x)\,\mathrm{d}x\,.$$

- The pace $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$ is separable, that is, there exists a countable dense subset in each $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$. In particular, the test function set $C_0^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$. However, the Banach space $L^\infty(\mathbb{R}^d)$ is not separable.

- The Banach space $L^p(\mathbb{R}^d)$, $p \in (1,\infty)$ is reflexive, and its dual space is isomorphic to $L^{p'}(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. However, $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ are not reflexiv: $(L^1)' = L^\infty$, $L^1 \subsetneq (L^\infty)'$.

- The Hölder's inequality holds: The product of two functions $f \in L^p$, $g \in L^{p'}$ lies in L^1, and furthermore

$$\int_{\mathbb{R}^d} |fg| \, \mathrm{d}x \leqslant \|f\|_{L^p} \|g\|_{L^{p'}}. \tag{1.2}$$

This implies immediately (noticing $|f|^q = (|f|^p)^{\frac{\theta q}{p}} (|f|^r)^{\frac{(1-\theta)q}{r}}$ and $1 = \frac{\theta q}{p} + \frac{(1-\theta)q}{r}$)

$$\|f\|_{L^q} \leqslant \|f\|_{L^p}^{\theta} \|f\|_{L^r}^{1-\theta} \quad \text{with } \frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}, \quad p \leqslant q \leqslant r, \quad \theta \in [0,1],$$
$$\text{and hence } \|f\|_{L^q} \leqslant \|f\|_{L^p} + \|f\|_{L^r}. \tag{1.3}$$

We recall briefly the proof of the fact that

$$C_c(\mathbb{R}^d) \subset L^p(\mathbb{R}^d), \quad p \in [1,\infty) \text{ densely}.$$

Since each complex-valued function f reads as $\mathrm{Re}\, f + i\mathrm{Im}\, f$, and each real-valued function f can be written as

$$f = f_+ - f_-,$$

where $f_+ = f 1_{\{f \geqslant 0\}}$ and $f_- = -f 1_{\{f<0\}}$ denote the positive and negative parts of f respectively, it suffices to consider nonnegative function f.

- Let $f \in L^1(\mathbb{R}^d)$ with $f \geqslant 0$. Since

$$\int_{\mathbb{R}^d} f dm^d = \int_0^\infty m^d(\{f > t\}) dt,$$

given $\varepsilon > 0$ there exists $0 = t_0 < t_1 < \ldots t_j < t_{j+1} < t_N < \infty$ so that (with $t_0 = 0$)

$$0 < \int_0^\infty m^d(\{f > t\}) dt - \sum_{j=1}^N (t_j - t_{j-1}) m^d(\{f > t_j\}) < \varepsilon.$$

Let

$$A_j = \{x : f(x) > t_j\}, \quad j = 1, \cdots, N,$$

such that

$$A_{j+1} \subset A_j \text{ and } m^d(A_1) < \infty.$$

Then

$$\Big\| f - \sum_{j=1}^{N} (t_j - t_{j-1})\chi_{A_j} \Big\|_{L^1} < \varepsilon.$$

Since there exists R such that $\int_{\{|x|>R\}} f dm^d < \varepsilon$, it suffices to approximate a characteristic function of a bounded measurable set A of finite measure by a continuous function. Let $\varepsilon > 0$. By inner and outer regularity of the Lebesgue measure, there exists a compact set K and an open set U so that

$$K \subset A \subset U \qquad m^d(U) < m^d(K) + \varepsilon.$$

Let $d(K, \mathbb{R}^d \backslash U) =: d_0 > 0$ and we define

$$f_L(x) = \max\{1 - Ld(x, K), 0\} \in C(\mathbb{R}^d).$$

If $d_0 L \geqslant 1$, then

$$\|f_L - \chi_A\|_{L^1} \leqslant \int_{U \backslash K} dm^d < \varepsilon.$$

If L is sufficiently large then $\operatorname{Supp} f_L$ is compact. Thus continuous functions with compact support are dense in $L^1(\mathbb{R}^d)$.

- If $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty)$ with $f \geqslant 0$, then $f^p \in L^1(\mathbb{R}^d)$, and as above we have $0 = t_0 < t_1 < \dots t_j < t_{j+1} < t_N < \infty$ so that (with $t_0 = 0$)

$$\begin{aligned} 0 &< \int_0^\infty m^d(\{f^p > t\})dt - \sum_{j=1}^{N} (t_j - t_{j-1}) m^d(\{f^p > t_j\}) \\ &= \int_{\mathbb{R}^d} \Big(f^p - \sum_{j=1}^{N-1} t_j \chi_{A_j \backslash A_{j+1}} - t_N \chi_{A_N} \Big) dx < \varepsilon. \end{aligned}$$

As $A_j \backslash A_{j+1}$ are disjoint measurable sets and $f > t_j$ on A_j, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \Big(f - \sum_{j=1}^{N-1} t_j^{1/p} \chi_{A_j \backslash A_{j+1}} - t_N^{1/p} \chi_{A_N} \Big)^p \mathrm{d}x \\ &\leqslant C_p \Big[\sum_{j=1}^{N-1} \int_{A_j \backslash A_{j+1}} \Big(f^p - t_j \Big) \mathrm{d}x + \int_{A_N} \Big(f^p - t_N \Big) \mathrm{d}x \Big] < C_p \varepsilon. \end{aligned}$$

As above, it suffices to approximate a characteristic function of a bounded measurable set A of finite measure by a continuous function with compact support in $L^p(\mathbb{R}^d)$. Indeed for L sufficiently large with $d_0L \geqslant 1$,

$$\|f_L - \chi_A\|_{L^p} \leqslant \varepsilon^{1/p}.$$

This density fact can also easily follow, provided that

- Any nonnegative measurable function can be approximated by a sequence of monotone nonnegative simple functions pointwise.
- Lebesgue dominate theorem.

Furthermore, we can take a mollifier, that is, a nonnegative function $\rho \in C_0^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \rho = 1$ (e.g. $\rho(x) = \left(\int_{B_1} e^{\frac{1}{|x|^2-1}}\right)^{-1} e^{\frac{1}{|x|^2-1}}$ for $|x| < 1$ and $= 0$ otherwise), and regularise a $L^p(\mathbb{R}^d)$-function as (**Exercise**)

$$\rho_n * f \to f \text{ in } L^p(\mathbb{R}^d), \quad p \in [1, \infty), \quad \rho_n(x) = n^d \rho(nx).$$

Since there exists $g \in C_c(\mathbb{R}^d)$ such that $\|g - f\|_{L^p} < \varepsilon$, we have

$$\|\rho_n * g - f\|_{L^p} < 2\varepsilon,$$

for large enough n. Since $\rho_n * g \in C_0^\infty(\mathbb{R}^d)$, this implies the density of $C_0^\infty(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$, $p \in [1, \infty)$. Since $C_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ (see also Proposition 1.2), we have

$$\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d), \quad p \in [1, \infty) \text{ densely}.$$

1.4.2 Fourier transform on $L^p(\mathbb{R}^d)$, $p \in [1, 2]$

Theorem 1.4. *The Fourier transform defines a unitary operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. In particular, the Fourier transform of $f \in L^2(\mathbb{R}^d)$ is a function in $L^2(\mathbb{R}^d)$ which reads as the following limit (almost everywhere)*

$$\mathcal{F}(f)(\xi) = \lim_{R\to\infty} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{|x|<R} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x\,. \tag{1.4}$$

Proof. Let $f \in L^2(\mathbb{R}^d)$ and let $(f_n)_n \subset \mathcal{S}(\mathbb{R}^d)$ converge to f in $L^2(\mathbb{R}^d)$. Then by Corollary 1.2, $\|\hat{f}_m - \hat{f}_n\|_{L^2} = \|f_m - f_n\|_{L^2}$ and hence $(\hat{f}_n)_n \subset \mathcal{S}(\mathbb{R}^d)$ is a Cauchy sequence in L^2. The unique limit of $(\hat{f}_n)_n$ in L^2 is the Fourier transform of f. The Fourier transform is a unitary operator, since $\int_{\mathbb{R}^d} f\bar{g} = \int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}}$ holds for $f, g \in \mathcal{S}(\mathbb{R}^d)$ and hence for $f, g \in L^2(\mathbb{R}^d)$. Since $f 1_{B_R(0)}$ converges to f in L^2-sense, the continuity of the Fourier transform implies the limit (1.4) in L^2-sense. □

For general $f \in L^p(\mathbb{R}^d)$, $p \in (1,2)$, we can always decompose it into two parts

$$f = f_1 + f_2 \text{ with } f_1 = f 1_{\{x \in \mathbb{R}^d \mid |f(x)| > 1\}} \text{ and } f_2 = f - f_1,$$

where

$$f_1 \in L^1(\mathbb{R}^d), \quad f_2 \in L^2(\mathbb{R}^d).$$

Therefore

$$\mathcal{F}(f) = \mathcal{F}(f_1) + \mathcal{F}(f_2) \in L^\infty(\mathbb{R}^d) + L^2(\mathbb{R}^d)$$

is a Lebesgue measurable function. Indeed by use of the Riesz-Thorin Interpolation theorem we have

Corollary 1.3 (Hausdorff-Young)**.** *If $f \in L^p(\mathbb{R}^d)$, $p \in [1,2]$, then $\mathcal{F}(f) \in L^{p'}(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, such that*

$$\|\mathcal{F}(f)\|_{L^{p'}(\mathbb{R}^d)} \leqslant (2\pi)^{-\frac{d}{2}(\frac{2}{p}-1)} \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof. Since the Fourier transform is a linear operator

$$\mathcal{F} : L^1(\mathbb{R}^d) \mapsto L^\infty(\mathbb{R}^d) \text{ with } \|\mathcal{F}\|_{L^1(\mathbb{R}^d) \mapsto L^\infty(\mathbb{R}^d)} \leqslant \frac{1}{(2\pi)^{\frac{d}{2}}},$$
$$\mathcal{F} : L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d) \text{ with } \|\mathcal{F}\|_{L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d)} = 1,$$

the result follows from Riesz-Thorin Interpolation theorem by virtue of

$$\frac{1}{p} = \frac{2(\frac{1}{p} - \frac{1}{2})}{1} + \frac{2(1 - \frac{1}{p})}{2}, \quad \frac{1}{p'} = \frac{2(\frac{1}{p} - \frac{1}{2})}{\infty} + \frac{2(1 - \frac{1}{p})}{2}.$$

□

Remark 1.6. *In general we don't know whether the Fourier transform of a $L^p(\mathbb{R}^d)$, $p > 2$-function still belongs to some Lebesgue space: We keep in mind that the Fourier transform of the constant function $f(x) = 1$ (which belongs to $L^\infty(\mathbb{R}^d)$) is a multiple of the Dirac-function.*

We recall the Young's inequality and investigate the Fourier transform of the product.

Lemma 1.2 (Young's inequality)**.** *If $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $p, q \in [1, \infty]$, then $f * g \in L^r(\mathbb{R}^d)$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, such that*

$$\|f * g\|_{L^r} \leqslant \|f\|_{L^p} \|g\|_{L^q}.$$

In particular, if $f, g \in \mathcal{S}(\mathbb{R}^d)$, then

$$\|\mathcal{F}(fg)\|_{L^r} \leqslant (2\pi)^{-\frac{d}{2}} \|\mathcal{F}(f)\|_{L^p} \|\mathcal{F}(g)\|_{L^q}.$$

Proof. Without loss of generality we consider $f, g \in \mathcal{S}(\mathbb{R}^d)$. It is trivial if one of p, q, r is ∞:

$$\|f * g\|_{L^\infty} \leqslant \|f\|_{L^p}\|g\|_{L^{p'}}.$$

We consider the case $p, q, r \in (1, \infty)$. Since $1 = \frac{p}{r} + \frac{p}{q'} = \frac{q}{r} + \frac{q}{p'}$, we have

$$\begin{aligned}
&\Big\|\int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y\Big\|_{L^r_x(\mathbb{R}^d)} \\
&\leqslant \Big\|\int_{\mathbb{R}^d}\Big(|f(x-y)|^{\frac{p}{r}}|g(y)|^{\frac{q}{r}}\Big)\Big(|f(x-y)|^{\frac{p}{q'}}\Big)\Big(|g(y)|^{\frac{q}{p'}}\Big)\,\mathrm{d}y\Big\|_{L^r_x(\mathbb{R}^d)}.
\end{aligned}$$

Noticing $1 = \frac{1}{r} + \frac{1}{p'} + \frac{1}{q'}$, we derive by Hölder's inequality that

$$\begin{aligned}
&\Big\|\int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y\Big\|_{L^r_x(\mathbb{R}^d)} \\
&\leqslant \Big\|\Big(\int_{\mathbb{R}^d}|f(x-y)|^p|g(y)|^q\,\mathrm{d}y\Big)^{\frac{1}{r}}\Big(\int_{\mathbb{R}^d}|f(x-y)|^p\,\mathrm{d}y\Big)^{\frac{1}{q'}}\Big(\int_{\mathbb{R}^d}|g(y)|^q\,\mathrm{d}y\Big)^{\frac{1}{p'}}\Big\|_{L^r_x(\mathbb{R}^d)} \\
&= \|f\|_{L^p}^{\frac{p}{q'}}\|g\|_{L^q}^{\frac{q}{p'}}\Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(x-y)|^p|g(y)|^q\,\mathrm{d}y\;\mathrm{d}x\Big)^{\frac{1}{r}} \\
&= \|f\|_{L^p}^{\frac{p}{q'}}\|g\|_{L^q}^{\frac{q}{p'}}\|f\|_{L^p}^{\frac{p}{r}}\|g\|_{L^q}^{\frac{q}{r}} = \|f\|_{L^p}\|g\|_{L^q}.
\end{aligned}$$

□

1.4.3 Bernstein-type inequalities

We have already seen from Proposition 1.4 that if a tempered distribution's Fourier transform is compacted supported at $\{0\}$, then it is a polynomial. We now investigate the relations between the $L^p(\mathbb{R}^d)$ functions and their derivatives when their Fourier transforms are compactly supported on a ball or on an annulus.

Lemma 1.3 (Bernstein)**.** *Let $\mathcal{B} = \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant R\}$ be a ball centered at 0 with radius $R > 0$ and $\mathcal{C} = \{\xi \in \mathbb{R}^d \mid 0 < R_1 \leqslant |\xi| \leqslant R_2\}$ be an annulus. Then there exists a constant C such that the following facts hold for any $k \in \mathbb{N}$, $\lambda > 0$, $p, q \in [1, \infty]$ with $p \leqslant q$ and $u \in L^p(\mathbb{R}^d)$:*

$$\operatorname{Supp}(\hat{u}) \subset \lambda\mathcal{B} \Rightarrow \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^q} \leqslant C^{k+1}\lambda^{k+\frac{d}{p}-\frac{d}{q}}\|u\|_{L^p}, \tag{1.5}$$

$$\operatorname{Supp}(\hat{u}) \subset \lambda\mathcal{C} \Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p} \leqslant \sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^p} \leqslant C^{k+1}\lambda^k\|u\|_{L^p}. \tag{1.6}$$

Proof. By scaling argument we can restrict ourselves to the case with $\lambda = 1$. Indeed, let

$$U(x) = \lambda^{-d}u(\lambda^{-1}x),$$

then (by Lemma 1.1)

$$\hat{U}(\xi) = \hat{u}(\lambda\xi),$$

and as $u(x) = \lambda^d U(\lambda x)$,

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} = \lambda^{d+k} \sup_{|\alpha|=k} \|(\partial^\alpha U)(\lambda\cdot)\|_{L^q} = \lambda^{d+k-\frac{d}{q}} \sup_{|\alpha|=k} \|\partial^\alpha U\|_{L^q},$$
$$\|u\|_{L^p} = \lambda^{d-\frac{d}{p}} \|U\|_{L^p}.$$

If $\mathrm{Supp}\,(\hat{u}) \subset \lambda\mathcal{B}$, then $\mathrm{Supp}\,(\hat{U}) \subset \mathcal{B}$. If (1.5) holds for U with Fourier transform supported in $\mathcal{B}$:

$$\sup_{|\alpha|=k} \|\partial^\alpha U\|_{L^q} \leqslant C^{k+1} \|U\|_{L^p},$$

then (1.5) holds for u with Fourier transform supported in $\lambda\mathcal{B}$. Similarly it suffices to prove (1.6) with $\lambda = 1$.
Let $\chi \in \mathcal{D}(\mathbb{R}^d)$ be a smooth cutoff function with value 1 on $\mathcal{B}$. If $\mathrm{Supp}\,(\hat{u}) \subset \mathcal{B}$, then

$$\hat{u} = \chi\hat{u} \Rightarrow u = (2\pi)^{-\frac{d}{2}}(\check{\chi} * u) \Rightarrow \partial^\alpha u = (2\pi)^{-\frac{d}{2}}(\partial^\alpha \check{\chi}) * u,$$

and hence by Young's inequality in Lemma 1.2,

$$\|\partial^\alpha u\|_{L^q} \leqslant (2\pi)^{-\frac{d}{2}} \|\partial^\alpha \check{\chi}\|_{L^r} \|u\|_{L^p}, \quad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}.$$

Obviously $\chi \in \mathcal{S}$ and $\partial^\alpha \check{\chi} \in \mathcal{S} \subset L^r$, while we can also estimate by virtue of (1.3)

$$\begin{aligned}\|f\|_{L^r} &\leqslant \|f\|_{L^1} + \|f\|_{L^\infty} \\ &\leqslant C_1 \|(1+|x|^2)^d f\|_{L^\infty} + \|f\|_{L^\infty} \\ &\leqslant C_2 \|(1-\Delta)^d \hat{f}\|_{L^1},\end{aligned}$$

we derive $\|\partial^\alpha \check{\chi}\|_{L^r} \leqslant C_2 \|(1-\Delta)^d(\xi^\alpha \chi)\|_{L^1} \leqslant C^{k+1}$ for any α with $|\alpha| = k$, such that $\|\partial^\alpha u\|_{L^q} \leqslant C^{k+1} \|u\|_{L^p}$ follows.
If $\mathrm{Supp}\,(\hat{u}) \subset \mathcal{C}$, then we can take $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ with $\varphi = 1$ on $\mathcal{C}$ such that

$$\hat{u} = \varphi\hat{u} = \varphi|\xi|^{-2k} \sum_{|\alpha|=k} (-i\xi)^\alpha (i\xi)^\alpha \hat{u} = \sum_{|\alpha|=k} \big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big) \widehat{\partial^\alpha u}.$$

[10.11.2021]

[17.11.2021]

Then by a similar argument as above we have

$$\|u\|_{L^p} \leqslant \sum_{|\alpha|=k} \Big\| \mathcal{F}^{-1}\big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big) * \partial^\alpha u \Big\|_{L^p} \leqslant \sum_{|\alpha|=k} \Big\| \mathcal{F}^{-1}\big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big) \Big\|_{L^1} \|\partial^\alpha u\|_{L^p},$$

where

$$\begin{aligned} \big\|\mathcal{F}^{-1}\big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big)\big\|_{L^1} &\leqslant C \big\|(1+|x|^2)^d \mathcal{F}^{-1}\big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big)\big\|_{L^\infty} \\ &\leqslant C \big\|(1-\Delta)^d \big(\varphi \frac{(-i\xi)^\alpha}{|\xi|^{2k}}\big)\big\|_{L^1} \leqslant C^{k+1}. \end{aligned}$$

□

Remark 1.7. *Any $L^p(\mathbb{R}^d)$-function with compactly supported Fourier transform is smooth.*

We also have the following lemma describing the action of the heat semigroup on L^p functions whose Fourier transform are supported on an annulus.

Lemma 1.4. *Let $\mathcal{C}$ be an annulus as in Lemma 1.3. Then there exists a constant C such that the following fact holds true for any $t > 0$, $\lambda > 0$, $p \in [1, \infty]$ and $u \in L^p$:*

$$\operatorname{Supp}(\hat{u}) \subset \lambda\mathcal{C} \Rightarrow \|e^{t\Delta}u\|_{L^q} \leqslant Ce^{-C^{-1}t\lambda^2} \lambda^{\frac{d}{p}-\frac{d}{q}} \|u\|_{L^p}, \quad \forall q \in [p, \infty],$$

where $e^{t\Delta}u$ is defined to be $\mathcal{F}^{-1}(e^{-t|\xi|^2}\hat{u}(\xi))$.

Proof. By scaling argument (**Exercise**) we can restrict ourselves to the case $\lambda = 1$. Take $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ as in the above proof, then if $\operatorname{Supp}(\hat{u}) \subset \mathcal{C}$,

$$\hat{u} = \varphi\hat{u} \Rightarrow \widehat{e^{t\Delta}u} = \varphi e^{-t|\xi|^2}\hat{u}.$$

As above, we can show similarly

$$\|\mathcal{F}^{-1}\big(\varphi e^{-t|\xi|^2}\big)\|_{L^r} \leqslant C\|(1-\Delta)^d(\varphi e^{-t|\xi|^2})\|_{L^1} \leqslant Ce^{-C^{-1}t}.$$

□

Remark 1.8. *If the Fourier transform of u is only compactly supported in a ball $\mathcal{B}$, then we do not have the exponential decay $e^{-C^{-1}t\lambda^2}$ (with respect to the time variable) in the inequality.*

It is straightforward to derive from Lemma 1.3 and Lemma 1.4 that

Corollary 1.4. *Let $\mathcal{C}$ be an annulus as above. Then there exists a constant C such that for any $T > 0$, $\lambda > 0$, $1 \leqslant p \leqslant q \leqslant \infty$, $1 \leqslant b \leqslant a \leqslant \infty$ there hold*

$$\mathrm{Supp}\,(\hat{u_0}) \subset \lambda\mathcal{C} \underset{u|_{t=0}=u_0}{\overset{(\partial_t-\mu\Delta)u=0}{\Longrightarrow}} \|u\|_{L^a([0,T];L^q)} \leqslant C(\mu\lambda^2)^{-\frac{1}{a}}\lambda^{\frac{d}{p}-\frac{d}{q}}\|u_0\|_{L^p}, \tag{1.7}$$

and

$$\begin{aligned}&\mathrm{Supp}\,(\hat{f}(t,\cdot)) \subset \lambda\mathcal{C},\ \forall t \in [0,T]\\ &\underset{u|_{t=0}=0}{\overset{(\partial_t-\mu\Delta)u=f}{\Longrightarrow}} \|u\|_{L^a([0,T];L^q)} \leqslant C(\mu\lambda^2)^{-1+\frac{1}{b}-\frac{1}{a}}\lambda^{\frac{d}{p}-\frac{d}{q}}\|f\|_{L^b([0,T];L^p)}.\end{aligned} \tag{1.8}$$

Proof. We take the Fourier transform of the Cauchy problem for the heat equation

$$\partial_t u - \mu\Delta u = 0, \quad u|_{t=0} = u_0,$$

to arrive at the following initial value problem of an ordinary differential equation

$$\partial_t \hat{u}(t,\xi) + \mu|\xi|^2\hat{u}(t,\xi) = 0, \quad \hat{u}(0,\xi) = \hat{u_0}(\xi),$$

for any fixed $\xi \in \mathbb{R}^d$. This ODE has a unique solution

$$\hat{u}(t,\xi) = e^{-\mu|\xi|^2 t}\hat{u_0}(\xi).$$

By Lemma 1.4, we have

$$\|u(t,\cdot)\|_{L^q} \leqslant Ce^{-C^{-1}\mu t\lambda^2}\lambda^{\frac{d}{p}-\frac{d}{q}}\|u_0\|_{L^p}.$$

We integrate with respect to the time variable to achieve

$$\|u\|_{L^a([0,T];L^q)} \leqslant C\|e^{-C^{-1}\mu t\lambda^2}\|_{L^a([0,T])}\lambda^{\frac{d}{p}-\frac{d}{q}}\|u_0\|_{L^p},$$

which implies (1.7) with a possibly larger constant C.
Similarly, the unique solution of the initial value problem

$$\partial_t u - \mu\Delta u = f, \quad u|_{t=0} = 0,$$

satisfies

$$\hat{u}(t,\xi) = \int_0^t e^{-\mu|\xi|^2(t-t')}\hat{f}(t',\xi)\,\mathrm{d}t'.$$

Lemma 1.4 implies then

$$\|u(t,\cdot)\|_{L^q} \leqslant C\int_0^t e^{-C^{-1}\mu(t-t')\lambda^2}\lambda^{\frac{d}{p}-\frac{d}{q}}\|f(t',\cdot)\|_{L^p}\,\mathrm{d}t' = Ce^{-C^{-1}\mu\lambda^2\cdot} *_t \|f(\cdot,x)\|_{L^p_x}.$$

We apply Young's inequality with respect to the time variable to arrive at (1.8), with a possibly larger constant C.

□

1.5 Stationary phase estimates

1.5.1 Van der Corput's lemma

We have the following interesting Van der Corput's Lemma in the one dimensional case.

Lemma 1.5 (Van der Corput's Lemma). *Let $(a, b) \subset \mathbb{R}$ be an arbitrary open interval. Let h be a real-valued measurable function. We define*

$$I(a, b) = \int_a^b e^{ih(t)} \, \mathrm{d}t \,.$$

Then

(1) If $|h'(t)| \geqslant \lambda > 0$ and h' is monotonic,

$$|I(a, b)| \leqslant C\lambda^{-1};$$

(2) If $h \in C^k([a, b])$, $k \geqslant 2$ and $|h^{(k)}| \geqslant \lambda > 0$,

$$|I(a, b)| \leqslant C\lambda^{-1/k}.$$

In either case the constants are independent of a and b (but may depend on k).

Proof. If $|h'(t)| \geqslant \lambda > 0$, then we can simply integrate by parts to get

$$\begin{aligned} I(a, b) &= \int_a^b ih'(t)e^{ih(t)} \frac{1}{ih'(t)} \, \mathrm{d}t \\ &= \int_a^b \frac{1}{ih'(t)} \, d(e^{ih(t)}) \\ &= \Big(\frac{e^{ih(\cdot)}}{ih'(\cdot)}\Big)\Big|_a^b - \int_a^b e^{ih(t)} \, d\Big(\frac{1}{ih'(t)}\Big). \end{aligned}$$

If h' is furthermore monotonic, we have

$$|I(a, b)| \leqslant \frac{2}{\lambda} + \int_a^b \Big| d\Big(\frac{1}{ih'(t)}\Big)\Big| = \frac{2}{\lambda} + \Big|\int_a^b d\Big(\frac{1}{ih'(t)}\Big)\Big| \leqslant \frac{4}{\lambda}.$$

For $k \geqslant 2$ the estimates follow by induction. If $h \in C^2([a, b])$ with $|h''| \geqslant \lambda > 0$, then without loss of generality we assume $h'' \geqslant \lambda > 0$. Hence h' is strictly increasing, and there exists at most one point $t_0 \in (a, b)$ such that $h'(t_0) = 0$.

We decompose the interval (a, b) into two parts (possibly with one part as an empty set):

$$(a, b) = J_1 \cup J_2, \quad J_1 = \{t \in (a, b) | |t - t_0| > \delta\},$$

with δ to be determined later. Then we have $|h'| \geqslant \lambda\delta$ on J_1 and we have by the above analysis

$$|\int_{J_1} e^{ih(t)}\, \mathrm{d}t\,| \leqslant \frac{8}{\lambda\delta}.$$

Since the length of J_2 is 2δ, we have

$$|\int_{J_2} e^{ih(t)}\, \mathrm{d}t\,| \leqslant 2\delta.$$

We take $\delta = 2\lambda^{-1/2}$ to arrive at

$$|I(a, b)| \leqslant \frac{8}{\lambda^{\frac{1}{2}}}.$$

For $k > 2$ we can similarly assume $h^{(k)} \geqslant \lambda > 0$ and consider the possible point t_0 with $h^{(k-1)}(t_0) = 0$. We finally take the parameter $\delta = \lambda^{-1/k}$. □

[17.11.2021]

[19.11.2021]

More generally we can use the above Lemma to show the following estimates **(Exercise)**.

Lemma 1.6 (Van der Corput's Lemma continued)**.** *Let h be a real-valued measurable function. Let $f \in W^{1,1}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) \,|\, f' \in L^1(\mathbb{R})\}$. We define*

$$I = \int_{\mathbb{R}} f(t) e^{ih(t)}\, \mathrm{d}t\,.$$

Then

(1) If $|h'(t)| \geqslant \lambda > 0$ and h' is monotonic,

$$|I| \leqslant C\lambda^{-1} \int_{\mathbb{R}} |f'|\, \mathrm{d}x\,;$$

(2) If $h \in C^k([a, b])$, $k \geqslant 2$ and $|h^{(k)}| \geqslant \lambda > 0$,

$$|I| \leqslant C\lambda^{-1/k} \int_{\mathbb{R}} |f'|\, \mathrm{d}x\,.$$

In either case the constants are independent of a and b.

We then derive easily from the above lemma 1.6 that for any $\xi \neq 0$,

$$|\hat{f}(\xi)| \leqslant C|\xi|^{-1}\|f'\|_{L^1(\mathbb{R})}.$$

This inequality can also be achieved easily from the observation $\widehat{f'} = i\xi\hat{f}$ in (1.1).

1.5.2 Higher dimensional case

We now consider the integral of the following form

$$I(\tau) = \int_{\mathbb{R}^d} e^{i\tau\Phi(\xi)}\psi(\xi)\,\mathrm{d}\xi\,,$$

where τ is a big parameter. We have seen in Lemma 1.6 that in the one dimensional case $d = 1$, for the nonstationary phase function Φ with monotonic derivative Φ' such that $|\Phi'| \geqslant c_0 > 0$, and for $\psi' \in L^1(\mathbb{R})$, we have the following decay estimate

$$|I(\tau)| \leqslant C\tau^{-1}.$$

We would like to generalize this estimate to higher dimensional case, and we will here for simplicity consider only the test function

$$\psi \in C_c^\infty(\mathbb{R}^d) \text{ with } \mathrm{Supp}\,(\psi) \subset K \text{ a given compact set in } \mathbb{R}^d, \tag{1.9}$$

and the real-valued smooth function

$$\Phi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}) \text{ with } \mathrm{Supp}\,(\psi) \subset K' \text{ a given compact set including } K. \tag{1.10}$$

Lemma 1.7 (Estimates for oscillatory integrals)**.** *Let ψ, Φ be test functions as in* (1.9)-(1.10)*. Let $c_0 \in (0,1)$ be a small positive constant. Then for any positive integers N, N' there exist $C_N, C_{N'}$ (depending only on N, N' and $\|\psi\|_{C^{\max\{N,N'\}}}, \|\partial^\alpha\Phi\|_{C^{\max\{N,N'\}-1}}$ with $|\alpha| = 2$) such that*

$$|I(\tau)| \leqslant \frac{C_N}{(c_0\tau)^N} + C_{N'}\int_{\mathbb{R}^d} \mathbf{1}_{\{\xi\in\mathbb{R}^d\,|\,|\nabla\Phi|\leqslant c_0\}}\frac{1}{(1+c_0\tau|\nabla\Phi|^2)^{N'}}\,\mathrm{d}\xi\,.$$

Proof. By changing Φ to Φ/c_0 and τ to $c_0\tau$ we can assume $c_0 = 1$ in the theorem (we will see in the following proof that the assumption is harmless). Let χ be a smooth cut-off function which is supported on the unit ball and takes the value 1 for $|x| \leqslant \frac{1}{2}$. We can decompose I into nonstationary phase part and stationary phase part:

$$I(\tau) = I_1(\tau) + I_2(\tau),$$

with

$$\begin{aligned} I_1(\tau) &= \int_{\mathbb{R}^d} e^{i\tau\Phi(\xi)} \Big(1 - \chi\big(\nabla\Phi(\xi)\big)\Big)\psi(\xi)\,\mathrm{d}\xi\,, \\ I_2(\tau) &= \int_{\mathbb{R}^d} e^{i\tau\Phi(\xi)}\,\chi\big(\nabla\Phi(\xi)\big)\psi(\xi)\,\mathrm{d}\xi\,. \end{aligned}$$

If $|\nabla\Phi(\xi)| \geqslant 1/2$, we observe the following fact

$$Le^{i\tau\Phi} = \tau e^{i\tau\Phi}, \text{ with the operator } L = \frac{1}{i}\sum_{j=1}^{d} \frac{\partial_{\xi_j}\Phi}{|\nabla\Phi|^2}\partial_{\xi_j}.$$

By integration by parts (as in the proof of Van der Corput's Lemma), we derive

$$I_1(\tau) = \frac{1}{\tau^N}\int_{\mathbb{R}^d} e^{i\tau\Phi}(L^t)^N\Big(\Big(1 - \chi\big(\nabla\Phi(\xi)\big)\Big)\psi(\xi)\Big)\,\mathrm{d}\xi\,,$$

where the operator

$$L^t = -L + i\frac{\Delta\Phi}{|\nabla\Phi|^2} - 2i\sum_{1\leqslant j,k\leqslant d}\frac{\partial_j\Phi\partial_k\Phi\partial_{jk}\Phi}{|\nabla\Phi|^4}.$$

As $(L^t)^N\Big(\Big(1 - \chi\big(\nabla\Phi(\xi)\big)\Big)\psi(\xi)\Big)$ is still test function supported in K, we have the estimate (**Exercise**)

$$|I_1(\tau)| \leqslant \frac{C_N}{\tau^N}.$$

If $|\nabla\Phi| \leqslant 1$, then we observe

$$L_\tau e^{i\tau\Phi} = e^{i\tau\Phi}, \text{ with the operator } L_t = \frac{1 + \frac{1}{i}\sum_{j=1}^{d}\partial_{\xi_j}\Phi\partial_{\xi_j}}{1 + \tau|\nabla\Phi|^2}.$$

By integration by parts again we derive

$$I_2(\tau) = \int_{\mathbb{R}^d} e^{i\tau\Phi}(L_\tau^t)^{N'}\Big(\chi\big(\nabla\Phi(\xi)\big)\psi(\xi)\Big)\,\mathrm{d}\xi\,,$$

where the operator

$$L_\tau^t = \frac{1 + i\sum_{j=1}^d \partial_{\xi_j}\Phi\partial_{\xi_j}}{1+\tau|\nabla\Phi|^2} + i\frac{\Delta\Phi}{1+\tau|\nabla\Phi|^2} - 2i\tau \sum_{1\leqslant j,k\leqslant d} \frac{\partial_j\Phi\partial_k\Phi\partial_{jk}\Phi}{(1+\tau|\nabla\Phi|^2)^2}.$$

Therefore we have the estimate (**Exercise**)

$$|I_2(\tau)| \leqslant C_{N'} \int_{\{|\nabla\Phi|\leqslant 1\}} \frac{1}{(1+\tau|\nabla\Phi|^2)^{N'}}\,\mathrm{d}\xi\,.$$

□

1.6 A dip on Fourier series

For the reason of completeness we introduce briefly the Fourier series for measurable periodic functions, and review the progress in the study of its convergence.

In his *Théorie analytique de la Chaleur (1822)*, J. Fourier arrived at the following problem:

Can we represent a function f (defined on an interval of $\mathbb{R}$) by a trigonometric series

$$f(x) = \sum_{k=0}^{\infty} a_k\cos(kx) + b_k\sin(kx)?$$

Or equivalently, using Euler's identity, does it hold

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}?$$

Obviously the function should be periodic with period 2π such that the representation could be true. We define the k-th Fourier coefficient of a function $f \in L^1([0,2\pi])$ by

$$\hat{f}(k) = \frac{1}{(2\pi)}\int_{[0,2\pi]} f(x)e^{-ik\cdot x}\,\mathrm{d}x\,,$$

and we call the following sequence $(\sigma_N f(x))_N$ the Fourier series of f:

$$\sigma_N f(x) = \sum_{|k|\leqslant N} \hat{f}(k)e^{ik\cdot x}, \quad N \in \mathbb{N}\,.$$

The above problem reads then as

Pointwise convergence problem: Does the limit $\lim_{N\to\infty}\sigma_N f$ *exist for each* x*,*

and if so, whether it is equal $f(x)$?
It is in general not true, and there exists a continuous function whose Fourier series diverges at a point *[Du Bois-Reymond 1873]*. We need some differentiability to ensure this pointwise convergence, and we recall the following two criteria in dimension one:

- Dini's criterion: If for some x there exists $\delta > 0$ such that

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| \mathrm{d}t < \infty,$$

  then

$$\lim_{N\to\infty} \sigma_N f(x) = f(x).$$

- Jordan's criterion: If f is a function of bounded variation in a neighborhood of x, then

$$\lim_{N\to\infty} \sigma_N f(x) = \frac{1}{2}(f(x_+) + f(x_-)).$$

In particular, if f is Hölder continuous near x: $|f(x+t) - f(x)| \leqslant C|t|^\alpha$, with $\alpha \in (0,1)$, then it is Dini-continuous and the above pointwise convergence result holds true.
We may also ask the above convergence problem in L^p-sense:
L^p-convergence problem: $\lim_{N\to\infty} \|\sigma_N f - f\|_{L^p} = 0$?
We consider first the whole line case $\mathbb{R}$ for a while: We define σ_R as follows:

$$\mathcal{F}(\sigma_R f) = 1_{(-R,R)} \mathcal{F}(f).$$

Then σ_R is related to the Hilbert transform:

$$Hf(x) := \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} \, \mathrm{d}y \,,$$

which is a bounded operator on $L^p(\mathbb{R})$, $p \in (1,\infty)$ (but not on $L^\infty(\mathbb{R})$, and weak $(1,1)$-type) as follows:

$$\sigma_R = 1_{(-R,R)}(D) = \frac{i}{2}(M_{-R} H M_R - M_R H M_{-R}), \text{ with } M_a f(x) = e^{iax} f(x).$$

Then σ_R is also a bounded operator on $L^p(\mathbb{R})$, $p \in (1,\infty)$ (which is indeed the convolution with the Dirichlet kernel). Similarly (but a little more complicated), the discrete version σ_N is also bounded on L^p, $p \in (1,\infty)$. Notice that for the trigonometric polynomial g, $\sigma_N g = g$ if $N \geqslant \deg(g)$. By the density of the triogonometric polynomials in L^p, $p \in [1,\infty)$, we derive

$$\lim_{N\to\infty} \|\sigma_N f - f\|_{L^p} = 0, \quad \forall p \in (1,\infty).$$

The endpoint cases $p = 1$ or ∞ do not hold: For $p = \infty$, we have already the classical counterexample of a continuous function by du Bois-Reymond. For $p = 1$, we have indeed

$$\|\sigma_N\|_{L^1 \mapsto L^1} = \frac{4}{\pi^2} \ln N + O(1).$$

In particular, we have the Parsevel's identity for $p = 2$:

$$\|f\|^2_{L^2([0,2\pi]^d)} = \sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2.$$

We can of course the higher dimensional Fourier series. We denote $L^p_{\text{per}}(\mathbb{R}^d)$, $p \in [1,\infty]$ to be the space of measurable functions $f : \mathbb{R}^d \mapsto \mathbb{C}$ which are periodic such that $f \in L^p([0,2\pi]^d)$. We identify the two $L^p_{\text{per}}(\mathbb{R}^d)$ functions if they are the same almost everywhere. We define the k, $k = (k_1, \cdots, k_d)$-th Fourier coefficient of a function $f \in L^1_{\text{per}}(\mathbb{R}^d)$ by

$$\hat{f}(k) = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} f(x) e^{-ik\cdot x} \, \mathrm{d}x \,,$$

and we call the following sequence $(\sigma_N f(x))_N$ the Fourier series of f:

$$\sigma_N f(x) = \sum_{|k|\leqslant N} \hat{f}(k) e^{ik\cdot x}, \quad N \in \mathbb{N}\,.$$

Intuitively, we go from sums to integrals (by considering T-periodic functions with $T \to \infty$), to realise the limit of the Fourier series as the inverse Fourier transform of its Fourier transform.

2 Littlewood-Paley theory

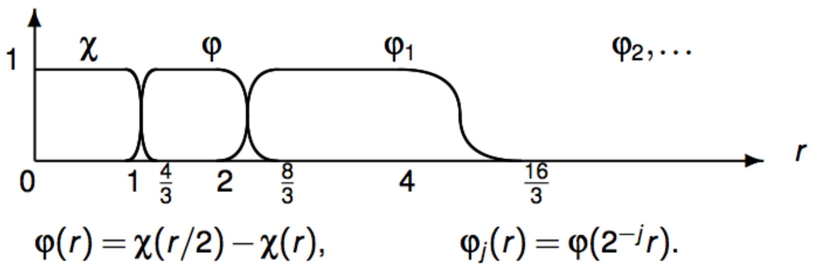
We have seen from Berstein's inequalities in Lemma 1.3 that if the Fourier transform $\hat{u}$ of a function $u \in L^p(\mathbb{R}^d)$ is compactly supported on an annulus, then the application of the derivatives ∇ on u works as a multiplication of λ on u: $\|\nabla u\|_{L^p} \sim \lambda \|u\|_{L^p}$, with λ denoting the size of Supp $(\hat{u})$.
We introduce the following dyadic partition of unity

$$1 = \chi(\xi) + \sum_{j \geqslant 0} \varphi_j(\xi), \quad \varphi_j(\xi) = \varphi(2^{-j}\xi) \quad \forall \xi \in \mathbb{R}^d, \tag{2.11}$$

where the radial functions

$$\chi \in \mathcal{D}(\mathcal{B}),\ \mathcal{B} = \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant \frac{4}{3}\},$$
$$\varphi = \chi(\cdot/2) - \chi \in \mathcal{D}(\mathcal{C}),\ \mathcal{C} = \{\xi \in \mathbb{R}^d \mid 1 \leqslant |\xi| \leqslant \frac{8}{3}\},$$

take the values in the interval $[0,1]$ as follows:



We can then do the Littlewood-Paley decomposition (formally) for $u \in L^p(\mathbb{R}^d)$ as follows:

$$u = \Delta_{-1}u + \sum_{j \geqslant 0} \Delta_j u, \text{ with } \widehat{\Delta_{-1}u} = \chi(\xi)\hat{u}(\xi), \quad \widehat{\Delta_j u} = \varphi_j(\xi)\hat{u}(\xi), \tag{2.12}$$

such that by Bernstein's inequalities

$$\|\Delta_j u\|_{L^q} \leqslant C 2^{j(\frac{d}{p} - \frac{d}{q})} \|\Delta_j u\|_{L^p}, \quad \forall j \geqslant -1, \quad \forall q \geqslant p,$$
$$\sup_{|\alpha|=k} \|\partial^\alpha \Delta_j u\|_{L^p} \geqslant C^{-(k+1)} 2^{jk} \|\Delta_j u\|_{L^p}, \quad \forall j \geqslant 0.$$

We also introduce the low-frequency cut-off operator S_j as follows:

$$S_j u = \sum_{j' \leqslant j-1} \Delta_{j'} u, \text{ i.e. } \widehat{S_j u} = \chi(2^{-j}\cdot)\hat{u}, \quad j \geqslant 0, \tag{2.13}$$

and we have

$$\|S_j u\|_{L^q} \leqslant C 2^{j(\frac{d}{p} - \frac{d}{q})} \|u\|_{L^p}, \quad \forall j \geqslant 0, \quad \forall q \geqslant p,$$

and $S_j \to \mathrm{Id}\,,\ j \to \infty$ on $\mathcal{S}'(\mathbb{R}^d)$:

[19.11.2021]

[24.11.2021]

Proposition 2.1. *Let* $u \in \mathcal{S}'(\mathbb{R}^d)$*, then*

$$S_j u \to u \text{ in } \mathcal{S}'(\mathbb{R}^d), \text{ as } j \to \infty.$$

Proof. **Exercise.** By duality and Theorem 1.2 it suffices to show

$$\widehat{S_j f} = \chi(2^{-j}\cdot)\hat{f} \to \hat{f} \text{ in } \mathcal{S}(\mathbb{R}^d).$$

□

Remark 2.1. *The Littlewood-Paley decomposition* (2.12)*:* $\mathrm{Id} = \sum_{j\geqslant -1}\Delta_j$ *is well-defined on* $\mathcal{S}'(\mathbb{R}^d)$*.*
If furthermore $u \in L^p(\mathbb{R}^d)$*,* $p \in [1,\infty]$*, then* $\Delta_j u, S_j u \in L^p(\mathbb{R}^d)$ *and for* $p \in [1,\infty)$ *we have (**Exercise**)*

$$S_j u \to u \text{ in } L^p(\mathbb{R}^d), \quad p \in [1,\infty).$$

2.1 Homogeneous Besov spaces

It is easy to notice from the dyadic partition of unity (2.11) that

$$1 = \sum_{j\in\mathbb{Z}} \varphi_j(\xi), \quad \forall \xi \neq 0.$$

We now restrict ourselves to the following tempered distributions whose Fourier transforms vanish at the origin in the following sense:

Definition 2.1. *We denote by* $\mathcal{S}'_h(\mathbb{R}^d)$ *the space of tempered distributions* $u \in \mathcal{S}'(\mathbb{R}^d)$ *such that*

$$\lim_{\lambda\to\infty} \|\theta(\lambda D)u\|_{L^\infty} = 0 \text{ with } \widehat{\theta(\lambda D)u} = \theta(\lambda\xi)\hat{u}(\xi), \quad \forall \theta \in \mathcal{D}(\mathbb{R}^d).$$

Remark 2.2. *Any tempered distribution* $u \in \mathcal{S}'$ *with* $\hat{u} \in L^1_{\mathrm{loc}}$ *belongs to* $\mathcal{S}'_h$*.*
Any L^p*,* $p \in [1,\infty)$ *function* u *belongs to* $\mathcal{S}'_h$ *since*

$$\|\theta(\lambda D)u\|_{L^\infty} = (2\pi)^{-\frac{d}{2}}\|\lambda^{-d}\check{\theta}(\lambda^{-1}\cdot) * u\|_{L^\infty} \leqslant \|u\|_{L^p}\|\lambda^{-d}\check{\theta}(\lambda^{-1}\cdot)\|_{L^{p'}} \to 0$$

as $\lambda \to \infty$*, as long as* $p' \neq 1$*.*
The polynomial $P \neq 0$ *does not belong to* $\mathcal{S}'_h$ *as* $\theta(\lambda\cdot)\hat{P} = \theta(0)\hat{P}$*.*

We then denote by $\dot\Delta_j$, $\dot S_j$, $j \in \mathbb{Z}$ the following operators (to distinguish from (2.12) and (2.13) where $j \in \mathbb{N}$):

$$\begin{aligned} \dot\Delta_j u &= \varphi(2^{-j}D)u \text{ with } \widehat{\varphi(2^{-j}D)u} = \varphi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z}, \\ \dot S_j u &= \chi(2^{-j}D)u \text{ with } \widehat{\chi(2^{-j}D)u} = \chi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z}, \end{aligned} \tag{2.14}$$

such that

$$u = \sum_{j\in\mathbb{Z}} \dot\Delta_j u \text{ and } \lim_{j\to-\infty} \dot S_j u = 0 \text{ in } \mathcal{S}', \text{ if } u \in \mathcal{S}'_h. \tag{2.15}$$

Then the (homogeneous) Littlewood-Paley decomposition: $\mathrm{Id} = \sum_{j\in\mathbb{Z}} \dot\Delta_j$ is well-defined in $\mathcal{S}'_h(\mathbb{R}^d)$.

2.1.1 Homogeneous Besov spaces

Definition 2.2 (Homogeneous Besov spaces). *Let $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$. We denote by $\dot B^s_{p,r}(\mathbb{R}^d)$ the set of tempered distributions $u \in \mathcal{S}'_h$ such that*

$$\|u\|_{\dot B^s_{p,r}} = \left\| \left(2^{js} \|\dot\Delta_j u\|_{L^p(\mathbb{R}^d)} \right)_{j\in\mathbb{Z}} \right\|_{\ell^r} < \infty. \tag{2.16}$$

It is easy to check that

$$L^p(\mathbb{R}^d) \subset \dot B^0_{p,\infty}(\mathbb{R}^d), \quad p \in [1, \infty).$$

Example 2.1. *The function $|x|^{-\sigma}$, $\sigma \in (0, d)$ belongs to $\dot B^{\frac{d}{p}-\sigma}_{p,\infty}$ for all $p \in [1, \infty]$, but not to $\dot B^{\frac{d}{p}-\sigma}_{p,r}$ for any $r \in [1, \infty)$.*

It suffices to show $|x|^{-\sigma} \in \dot B^{d-\sigma}_{1,\infty}$ for $\sigma \in (0, d)$, such that by Proposition 2.2 (see below), $|x|^{-\sigma} \in \dot B^{d-\sigma}_{1,\infty} \subset \dot B^{\frac{d}{p}-\sigma}_{p,\infty}$ for any $p \in [1, \infty]$.
Exercise (Hint: Observe that $\dot\Delta_j f = 2^{j\sigma}(\dot\Delta_0 f)(2^j \cdot)$ and hence $\|\dot\Delta_j f\|_{L^1} = 2^{j(\sigma-d)}\|\dot\Delta_0 f\|_{L^1}$.)

Proposition 2.2 (Basic properties of $\dot B^s_{p,r}(\mathbb{R}^d)$). *We have the following basic properties for the homogeneous Besov spaces:*

- *Homogeneity:* $\|u(\lambda\cdot)\|_{\dot B^s_{p,r}} = \lambda^{s-\frac{d}{p}}\|u\|_{\dot B^s_{p,r}}$, $\forall \lambda \in 2^{\mathbb{N}}$;
- *Embedding:* $\dot B^s_{p,r} \hookrightarrow \dot B^{s-d(\frac{1}{p}-\frac{1}{p_1})}_{p_1,r_1}$, *if $p \leqslant p_1$, $r \leqslant r_1$ and in particular* $\dot B^s_{p,1} \hookrightarrow \dot B^s_{p,r} \hookrightarrow \dot B^s_{p,\infty}$;

- *Interpolation:* $\dot B^{s_1}_{p,r}\cap \dot B^{s_2}_{p,r}\hookrightarrow \dot B^{s}_{p,r}$, $\forall s\in[s_1,s_2]$ *and* $\dot B^{s_1}_{p,\infty}\cap \dot B^{s_2}_{p,\infty}\hookrightarrow \dot B^{s}_{p,1}$, $\forall s\in(s_1,s_2)$. *Furthermore, we have for any* $u\in\mathcal{S}'_h$,

$$\begin{aligned}\|u\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,r}} &\leqslant \|u\|^{\theta}_{\dot B^{s_1}_{p,r}}\|u\|^{1-\theta}_{\dot B^{s_2}_{p,r}}, \quad \forall\theta\in[0,1];\\ \|u\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,1}} &\leqslant \tfrac{C}{s_2-s_1}\big(\tfrac{1}{\theta}+\tfrac{1}{1-\theta}\big)\|u\|^{\theta}_{\dot B^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{\dot B^{s_2}_{p,\infty}}, \quad \forall\theta\in(0,1).\end{aligned} \tag{2.17}$$

[24.11.2021]

[01.12.2021]

Proof. The homogeneity property follows easily from the change of variables (**Exercise**).
The embedding results follow straighforward from Bernstein's inequalities in Lemma 1.3 (**Exercise**).
The first inequality in (2.17) follows easily from

$$2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} = \big(2^{js_1}\|\dot\Delta_j u\|_{L^p}\big)^{\theta}\big(2^{js_2}\|\dot\Delta_j u\|_{L^p}\big)^{1-\theta}$$

and Hölder's inequality for the $\|\|_{\ell^r}$-norm:

$$\|(c_j\cdot d_j)_{j\in\mathbb{Z}}\|_{\ell^r}\leqslant \|(c_j)_{j\in\mathbb{Z}}\|_{\ell^{r_1}}\|(d_j)_{j\in\mathbb{Z}}\|_{\ell^{r_2}}, \quad \text{with } \frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2},\ r,r_1,r_2\in[1,\infty].$$

(In particular we notice here $\frac{1}{r}=\frac{1}{r/\theta}+\frac{1}{r/(1-\theta)}$).
In order to show the second inequality in (2.17), we separate the high frequency and low frequency part in the definition of Besov norms:

$$\|u\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{p,1}} = \sum_{j\leqslant N}2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p}+\sum_{j>N}2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p},$$

with the frequency threshold 2^N to be determined later. We use the Besov norm $\dot B^{s_1}_{p,\infty}$ to control the low frequency part:

$$\begin{aligned}\sum_{j\leqslant N}2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p} &\leqslant \sum_{j\leqslant N}2^{j(\theta s_1+(1-\theta)s_2)}\big(2^{-js_1}\|u\|_{\dot B^{s_1}_{p,\infty}}\big)\\ &= \sum_{j\leqslant N}2^{j(1-\theta)(s_2-s_1)}\|u\|_{\dot B^{s_1}_{p,\infty}} = \frac{2^{N(1-\theta)(s_2-s_1)}}{2^{(1-\theta)(s_2-s_1)}-1}\|u\|_{\dot B^{s_1}_{p,\infty}},\end{aligned}$$

and similarly we use the Besov norm $\dot B^{s_2}_{p,\infty}$ to control the high frequency part:

$$\sum_{j>N}2^{j(\theta s_1+(1-\theta)s_2)}\|\dot\Delta_j u\|_{L^p}\leqslant \sum_{j>N}2^{j(\theta s_1+(1-\theta)s_2)}\big(2^{-js_2}\|u\|_{\dot B^{s_2}_{p,\infty}}\big)$$

$$= \sum_{j>N} 2^{-j\theta(s_2-s_1)} \|u\|_{\dot{B}^{s_2}_{p,\infty}} \leqslant \frac{2^{-N\theta(s_2-s_1)}}{1-2^{-\theta(s_2-s_1)}} \|u\|_{\dot{B}^{s_2}_{p,\infty}}.$$

We choose N such that

$$2^{N(s_2-s_1)} \sim \frac{\|u\|_{\dot{B}^{s_2}_{p,\infty}}}{\|u\|_{\dot{B}^{s_1}_{p,\infty}}} \text{ e.g. } N = \text{integer part of } \frac{1}{s_2-s_1} \log_2\Big(\frac{\|u\|_{\dot{B}^{s_2}_{p,\infty}}}{\|u\|_{\dot{B}^{s_1}_{p,\infty}}}\Big),$$

such that the second inequality in (2.17) holds. □

We have the following facts for the homogeneous Besov spaces.

Theorem 2.1 (Homogeneous Besov spaces). *(i) The definition of the homogeneous Besov space $\dot{B}^s_{p,r}$ is independent of the choice of the function φ in the dyadic partition.*

Furthermore, let $\mathcal{C}'$ be an annulus and $(u_j)_{j\in\mathbb{Z}}$ be a sequence of functions such that

$$\mathrm{Supp}\,(\hat{u_j}) \subset 2^j\mathcal{C}', \quad \big(2^{js}\|u_j\|_{L^p}\big)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z}), \quad \sum_{j\in\mathbb{Z}} u_j \to u \text{ in } \mathcal{S}' \text{ with } u \in \mathcal{S}'_h,$$

then $u \in \dot{B}^s_{p,r}$ and

$$\|u\|_{\dot{B}^s_{p,r}} \leqslant C(s) \Big\| \big(2^{js}\|u_j\|_{L^p}\big)_{j\in\mathbb{Z}} \Big\|_{\ell^r}.$$

If furthermore $s>0$, then for the sequence of functions $(v_j)_{j\in\mathbb{Z}}$ satisfying (with some ball $\mathcal{B}'$)

$$\mathrm{Supp}\,(\hat{v_j}) \subset 2^j\mathcal{B}', \quad \big(2^{js}\|v_j\|_{L^p}\big)_{j\in\mathbb{Z}} \in \ell^r(\mathbb{Z}), \quad \sum_{j\in\mathbb{Z}} v_j \to v \text{ in } \mathcal{S}' \text{ with } v \in \mathcal{S}'_h,$$

we have $v \in \dot{B}^s_{p,r}$ and

$$\|v\|_{\dot{B}^s_{p,r}} \leqslant C(s) \Big\| \big(2^{js}\|v_j\|_{L^p}\big)_{j\in\mathbb{Z}} \Big\|_{\ell^r}.$$

(ii) The homogeneous Besov space $\dot{B}^s_{p,r} \subset \mathcal{S}'_h$, $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$ is a normed space. If

$$s < \frac{d}{p} \text{ or } (s,p,r) = (\frac{d}{p}, p, 1), \tag{$*$}$$

then $\dot{B}^s_{p,r}(\mathbb{R}^d)$ is a Banach space. In particular, we have the following Fatou's property: If (U_n) is a bounded sequence of $\dot{B}^s_{p,r}(\mathbb{R}^d)$ satisfying $(*)$, then there exist $u \in \dot{B}^s_{p,r}(\mathbb{R}^d)$ and a subsequence (U_{n_k}) such that

$$\lim_{n_k\to\infty} U_{n_k} = u \text{ in } \mathcal{S}' \text{ and } \|u\|_{\dot{B}^s_{p,r}} \leqslant C \liminf_{n_k\to\infty} \|U_{n_k}\|_{\dot{B}^s_{p,r}}.$$

(iii) If $(p,r) \in [1,\infty)^2$, then the space $\mathcal{S}_0(\mathbb{R}^d) := \{f \in \mathcal{S}(\mathbb{R}^d) \,|\, \mathrm{Supp}\,(\hat{f}) \cap \{0\} = \varnothing\}$ is dense in $\dot{B}^s_{p,r}(\mathbb{R}^d)$. In particular, the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is dense in $\dot{B}^s_{p,r}(\mathbb{R}^d)$, with $p < \infty$ and $r < \infty$.

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[03.12.2021]

Proof. Proof of (i): If $\mathrm{Supp}\,(\hat{u_j}) \subset 2^j\mathcal{C}'$, then there exists $J \in \mathbb{N}$ such that

$$u_j = \sum_{|k-j|\leqslant J} \dot{\Delta}_k u_j \text{ and hence } \dot{\Delta}_k u = \sum_{|j-k|\leqslant J} \dot{\Delta}_k u_j.$$

Thus

$$\big\|(2^{ks}\|\dot{\Delta}_k u\|_{L^p})\big\|_{\ell^r} \leqslant \big\|(\sum_{|j-k|\leqslant J} 2^{(k-j)s}2^{js}\|u_j\|_{L^p})_k\big\|_{\ell^r} \leqslant C(J,s)\big\|(2^{js}\|u_j\|_{L^p})_j\big\|_{\ell^r}.$$

If $\mathrm{Supp}\,(\hat{v_j}) \subset 2^j\mathcal{B}$, then there exists $J \in \mathbb{N}$ such that $\dot{\Delta}_k v = \sum_{j\geqslant k-J} \dot{\Delta}_k v_j$ and hence

$$\big\|(2^{ks}\|\dot{\Delta}_k v\|_{L^p})\big\|_{\ell^r} \leqslant \big\|(\sum_{j\geqslant k-J} 2^{(k-j)s}2^{js}\|v_j\|_{L^p})_k\big\|_{\ell^r} \leqslant C(J,s)\big\|(2^{js}\|v_j\|_{L^p})_j\big\|_{\ell^r},$$

where the last inequality is ensured by $s > 0$.
Similarly, in order to show the independence of the choice of the function φ in the dyadic partition in the definition of the homogeneous Besov space $\dot{B}^s_{p,r}$, we take another function $\tilde{\varphi}$ in the dyadic partition. Then there exists $J \in \mathbb{N}$ such that $\mathrm{Supp}\,\varphi(2^{-j}\cdot) \cap \mathrm{Supp}\,\tilde{\varphi}(2^{-j'}\cdot) = \varnothing$ if $|j-j'| \geqslant J$ and hence the norms $\|u\|_{\dot{B}^s_{p,r}}$ and $\|u\|_{\widetilde{\dot{B}^s_{p,r}}}$ are equivalent.

Proof of (ii): By view of (2.15), the seminorm $\|\cdot\|_{\dot{B}^s_{p,r}}$ is indeed a norm in the vector space $\dot{B}^s_{p,r}(\mathbb{R}^d)$.
We claim that if $u \in \mathcal{S}'$ such that $\|u\|_{\dot{B}^s_{p,r}} < \infty$ with (s,p,r) satisfying $(*)$, then $u \in \mathcal{S}'_h$ and thus $u \in \dot{B}^s_{p,r}(\mathbb{R}^d)$. Indeed, it suffices to show $\|\dot{S}_j u\|_{L^\infty} \to 0$. By Lemma 1.3, it holds that

$$\|\dot{\Delta}_{j'} u\|_{L^\infty} \leqslant 2^{j'\frac{d}{p}}\|\dot{\Delta}_{j'} u\|_{L^p} = 2^{j'(\frac{d}{p}-s)}\big(2^{j's}\|\dot{\Delta}_{j'} u\|_{L^p}\big).$$

If $s < \frac{d}{p}$, then

$$\lim_{j\to-\infty} \sum_{j'<j} \|\dot{\Delta}_{j'} u\|_{L^\infty} \leqslant C \lim_{j\to-\infty} 2^{j(\frac{d}{p}-s)} \sup{}_{j'}\big(2^{j's}\|\dot{\Delta}_{j'} u\|_{L^p}\big) = 0.$$

If $s=\frac{d}{p}$ and $r=1$, then

$$\lim_{j\to-\infty}\sum_{j'<j}\|\dot\Delta_{j'}u\|_{L^\infty}\leqslant\lim_{j\to-\infty}\sum_{j'<j}\big(2^{j's}\|\dot\Delta_{j'}u\|_{L^p}\big)=0.$$

Hence $\|\dot S_j u\|_{L^\infty}\to 0$, and thus $u=\sum_{j\in\mathbb{Z}}\dot\Delta_j u\in\dot B^s_{p,r}(\mathbb{R}^d)\subset\mathcal{S}'_h$ if $s<\frac{d}{p}$ or $(s,p,r)=(\frac{d}{p},p,1)$.
In order to show the Fatou's property, we first use Cantor's diagonal process (noticing that for each $j\in\mathbb{Z}$, the sequence $\dot\Delta_j U_n$ is uniformly bounded in $L^p\cap L^\infty$) to find a subsequence (U_{n_k}) as well as a sequence of smooth functions (u_j) whose Fourier transform is supported in $2^j\mathcal{C}$ such that

$$\dot\Delta_j U_{n_k}\to u_j \text{ in } \mathcal{S}' \text{ and } \big\|(2^{js}\|u_j\|_{L^p})\big\|_{\ell^r}\leqslant\big\|(2^{js}\liminf_{n_k}\|\dot\Delta_j U_{n_k}\|_{L^p})\big\|_{\ell^r}<\infty.$$

Thus $\sum_j u_j$ converges to some limit u in $\mathcal{S}'$ with $\|u\|_{\dot B^s_{p,r}}<\infty$ (by virtue of the proof of (i) above). If (s,p,r) satisfy $(*)$, then $u\in\mathcal{S}'_h$, and indeed in $\dot B^s_{p,r}(\mathbb{R}^d)$ by the above claim. We can further show that U_{n_k} converges to u in $\mathcal{S}'$: Firstly, for any fixed $j\in\mathbb{Z}$ it holds

$$\dot\Delta_j u=\sum_{|j'-j|\leqslant 1}\dot\Delta_j u_{j'} \text{ with } u_{j'}=\lim_{n_k\to\infty}\dot\Delta_{j'}U_{n_k} \text{ in } \mathcal{S}',$$

and hence for any fixed M,N with $M<N$ it holds

$$\sum_{j=M}^N\dot\Delta_j u=\lim_{n_k\to\infty}\sum_{j=M}^N\dot\Delta_j U_{n_k} \text{ in } \mathcal{S}'.$$

Secondly if $(*)$ holds, then $\dot S_M U_{n_k}\to 0$ as $M\to-\infty$ uniformly (by use of Bernstein's inequality). Similarly, $(\mathrm{Id}-\dot S_N)U_{n_k}\to 0$ as $N\to\infty$ uniformly, say in $\dot B^{s-1}_{p,r}$.
By use of Fatou's property we can show the convergence of a Cauchy sequence in $\dot B^s_{p,r}(\mathbb{R}^d)$ satisfying $(*)$ (**Exercise**).

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Proof of (iii): In order to show the density result in (ii), for any $u\in\dot B^s_{p,r}$, $p,r<\infty$, for any $\varepsilon>0$, we take the approximated function

$$u^R_{M,N}=(\mathrm{Id}-\dot S_{-M})\big(\chi(\frac{\cdot}{R})u_N\big),\quad u_N:=\sum_{|j|\leqslant N}\dot\Delta_j u,$$

where N is chosen such that (as $r<\infty$)

$$\|u-u_N\|_{\dot B^s_{p,r}}<\frac{\varepsilon}{2}.$$

We take $M > N$ and the parameter R will be determined later. Then $u^R_{M,N} \in \mathcal{S}_0$ (while $u_N \in W^{\infty,p}(\mathbb{R}^d)$ is not necessarily in $\mathcal{S}$), and as $M > N$, we calculate

$$\begin{aligned}\|u^R_{M,N} - u_N\|_{\dot B^s_{p,r}} &= \Big\|(\mathrm{Id} - \dot S_{-M})\big(\chi(\frac{\cdot}{R}) - 1\big)u_N\Big\|_{\dot B^s_{p,r}} \\ &= \Big(\sum_{j\geqslant -M-1} 2^{jsr}\|\dot\Delta_j\big(\big(\chi(\frac{\cdot}{R}) - 1\big)u_N\big)\|^r_{L^p}\Big)^{\frac{1}{r}}.\end{aligned}$$

We decompose the summation $\sum_{j\geqslant -M-1}$ into high and low frequency parts separately again, such that

$$\begin{aligned}\sum_{j\geqslant 0} 2^{js}\|\dot\Delta_j\big(\big(\chi(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p} &\leqslant C\sup{}_{j\geqslant 0}2^{j([s]+2)}\|\dot\Delta_j\big(\big(\chi(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p} \\ &\leqslant C\sup{}_{|\alpha|=[s]+2}\|\partial^\alpha\big(\big(\chi(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p}, \\ \sum_{j=-M-1}^{-1} 2^{js}\|\dot\Delta_j\big(\big(\theta(\frac{\cdot}{R}) - 1\big)u_N\big)\|_{L^p} &\leqslant C(M,s)\|\big(\chi(\frac{\cdot}{R}) - 1\big)u_N\|_{L^p},\end{aligned}$$

and therefore for fixed M, N, as $p < \infty$, we can choose R large enough such that (as $p < \infty$)

$$\|u^R_{M,N} - u_N\|_{\dot B^s_{p,r}} < \frac{\varepsilon}{2},$$

and hence $\|u^R_{M,N} - u\|_{\dot B^s_{p,r}} < \varepsilon$.

□

Remark 2.3. *If s, p, r does not satisfy $(*)$, then due to the possible infrared divergence in the low frequency part (keeping in mind the polynomials), $\dot B^s_{p,r}(\mathbb{R}^d)$ is not a Banach space.*
We have the following more general Fatou's property. The space $\dot B^s_{p,r} \cap \dot B^{s_1}_{p_1,r_1}$ with (s,p,r) satisfying $(*)$ endowed with the norm $\|\|_{\dot B^s_{p,r}} + \|\|_{\dot B^{s_1}_{p_1,r_1}}$ is complete and satisfies the Fatou property: If (U_n) is a bounded sequence of $\dot B^s_{p,r}(\mathbb{R}^d) \cap \dot B^{s_1}_{p_1,r_1}$ satisfying $(*)$, then there exist $u \in \dot B^s_{p,r}(\mathbb{R}^d) \cap \dot B^{s_1}_{p_1,r_1}$ and a subsequence (U_{n_k}) such that

$$\lim_{n_k\to\infty} U_{n_k} = u \text{ in } \mathcal{S}',\ \|u\|_{\dot B^s_{p,r}} \leqslant C\liminf_{n_k\to\infty}\|U_{n_k}\|_{\dot B^s_{p,r}},\ \|u\|_{\dot B^{s_1}_{p_1,r_1}} \leqslant C\liminf_{n_k\to\infty}\|U_{n_k}\|_{\dot B^{s_1}_{p_1,r_1}}.$$

2.1.2 Sobolev spaces and Lebesgue spaces

We define the homogeneous Sobolev spaces $\dot H^s$ as follows:

Definition 2.3 (Homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^d)$)**.** *Let $s \in \mathbb{R}$. We denote by $\dot{H}^s(\mathbb{R}^d)$ the set of tempered distributions $u \in \mathcal{S}'$ such that $\hat{u} \in L^1_{\text{loc}}(\mathbb{R}^d)$ and*

$$\|u\|_{\dot{H}^s} = \Big(\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2 \,\mathrm{d}\xi\Big)^{\frac{1}{2}} < \infty.$$

Then $\dot{H}^s \subset \mathcal{S}'_h$. We have the following facts (**Exercise**)

- Since $\|\|_{\dot{H}^s} \sim \|\|_{\dot{B}^s_{2,2}}$, $\dot{H}^s \subset \dot{B}^s_{2,2}$ for any $s \in \mathbb{R}$.
- If $s < \frac{d}{2}$ (i.e. $(*)$ holds), then $\dot{H}^s = \dot{B}^s_{2,2}$.

We also have the following relations between the Besov spaces and the Lebesgue spaces:

Proposition 2.3 (Relations between Besov spaces and Lebesgue spaces)**.** *Let $(p,q) \in [1,\infty]^2$. Then*

- $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$ *and more generally* $\dot{B}^{\frac{d}{p}-\frac{d}{q}}_{p,1} \hookrightarrow L^q$ *whenever* $p \leqslant q$*;*
- $\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$ *whenever* $p \in [1,\infty)$*;*
- *The space of bounded measures on $\mathbb{R}^d$ is continuously embedded in $\dot{B}^0_{1,\infty}$.*

Proof. **Exercise.** □

Remark 2.4. *We also have the following facts: $\dot{B}^0_{p,2} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,p}$ whenever $p \in [2,\infty)$ and $\dot{B}^0_{p,p} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,2}$ whenever $p \in (1,2]$, and in particular $L^2 = \dot{B}^0_{2,2}$; $\|u\|_{L^q} \leqslant C\|u\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}}\|u\|^{\theta}_{\dot{B}^{\beta}_{p,p}}$ whenever $1 \leqslant p < q < \infty$, $\alpha \in \mathbb{R}^+$ with $\theta = \frac{p}{q}$ and $\beta = \alpha(\frac{q}{p}-1)$.*
We may also define the homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,r}(\mathbb{R}^d)$, $1 < p < \infty$ as

$$\dot{F}^s_{p,r}(\mathbb{R}^d) = \{u \in \mathcal{S}'_h \,|\, \|u\|_{\dot{F}^s_{p,r}} = \Big\|\big\|\big(2^{js}|\dot{\Delta}_j u(x)|\big)_{j\in\mathbb{Z}}\big\|_{\ell^r}\Big\|_{L^p} < \infty\},$$

and the Lorentz space $L^{p,r}(\mathbb{R}^d)$, $1 \leqslant p < \infty$ as

$$L^{p,r}(\mathbb{R}^d) = \{u : \mathbb{R}^d \mapsto \mathbb{C} \,|\, \|u\|_{L^{p,r}} = \big\|s^{\frac{1}{r}}u^*(s)\big\|_{L^r(\mathbb{R}^+;\frac{ds}{s})} < \infty\},$$

where u^* is the rearrangement function of u:

$$u^*(s) = \inf\{\lambda \,|\, m(\{x \in \mathbb{R}^d \,|\, |u(x)| \geqslant \lambda\}) \leqslant s\}.$$

There are relations among the above mentioned functional spaces which we do not go to details here.

2.1.3 Homogeneous paradifferential calculus

Let $u, v \in \mathcal{S}'_h$ and we would like to consider their product uv (which is in general not well defined). Recall the Littlewood-Paley decomposition (2.14)-(2.15):

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \text{ and } \lim_{j\to-\infty} \dot{S}_j u = 0 \text{ in } \mathcal{S}', \quad \lim_{j\to+\infty} \dot{S}_j u = u \text{ in } \mathcal{S}',$$
$$v = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j v \text{ and } \lim_{j\to-\infty} \dot{S}_j v = 0 \text{ in } \mathcal{S}', \quad \lim_{j\to+\infty} \dot{S}_j v = v \text{ in } \mathcal{S}',$$

where

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u \text{ with } \widehat{\varphi(2^{-j}D)u} = \varphi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z},$$
$$\dot{S}_j u = \chi(2^{-j}D)u \text{ with } \widehat{\chi(2^{-j}D)u} = \chi(2^{-j}\xi)\hat{u}(\xi), \quad j \in \mathbb{Z}\,.$$

Then *formally* we can write the product uv as

$$uv = \sum_{j,k\in\mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}_k v,$$

and we can split the above sum into three parts:

$$uv = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v + \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \dot{S}_{j-1} v + \sum_{|j-k|\leqslant 1} \dot{\Delta}_j u \dot{\Delta}_k v.$$

[08.12.2021]

[15.12.2021]

Notice that the Fourier transform of $\dot{S}_{j-1}u = \chi(2^{-(j-1)}D)u$ is compactly supported on the ball of size $\frac{4}{3}2^{(j-1)} = \frac{2}{3}2^j$ and the Fourier transform of $\dot{\Delta}_j u = \varphi(2^{-j}D)u$ is compactly supported on the annulus $\{\xi \in \mathbb{R}^d \mid 2^j \leqslant |\xi| \leqslant \frac{8}{3}2^j\}$, such that

$$\widehat{\dot{S}_{j-1}u\dot{\Delta}_j v} = (2\pi)^{-\frac{d}{2}} \widehat{\dot{S}_{j-1}u} * \widehat{\dot{\Delta}_j v}, \quad j \in \mathbb{Z}$$

is compactly supported on the annulus $\{\xi \in \mathbb{R}^d \mid \frac{1}{3}2^j \leqslant |\xi| \leqslant \frac{10}{3}2^j\}$, while

$$\widehat{\dot{\Delta}_j u \dot{\Delta}_k v} = (2\pi)^{-\frac{d}{2}} \widehat{\dot{\Delta}_j u} * \widehat{\dot{\Delta}_k v}, \quad |j-k| \leqslant 1$$

is compactly supported on the ball of size $8 \cdot 2^j$.
Notations Let $u, v \in \mathcal{S}'_h$. We denote by $\dot{T}_u v$ the homogeneous paraproduct of v by u as follows:

$$\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v.$$

We denote by $\dot{R}(u,v)$ the homogeneous remainder of u and v as follows:

$$\dot{R}(u,v) = \sum_{|j-k|\leqslant 1} \dot{\Delta}_j u \dot{\Delta}_k v.$$

We call the above product decomposition of uv the Bony decomposition:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v).$$

The summations in the above "definitions" of the bilinear operators $\dot{T}, \dot{R}$ are formal and we are going to make it rigorous in the setting of homogeneous Besov spaces, by use of the above consideration of the supports of the Fourier transforms and Theorem 2.1.

Theorem 2.2 (Estimates for the products)**.** *We have the following estimates for the paraproducts and the remainder term:*

(i) Let $(s,p,r) \in \mathbb{R} \times [1,\infty]^2$ satisfy $(*)$: $s < \frac{d}{p}$ or $(s,p,r) = (\frac{d}{p}, p, 1)$. Then

- *For $u \in L^\infty(\mathbb{R}^d)$ and $v \in \dot{B}^s_{p,r}(\mathbb{R}^d)$, $\dot{T}_u v \in \dot{B}^s_{p,r}$ and there exists C (depending only on s) such that*

$$\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \leqslant C\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}}.$$

- *For $u \in \dot{B}^\sigma_{\infty,r_1}(\mathbb{R}^d)$, $\sigma < 0$ and $v \in \dot{B}^s_{p,r_2}(\mathbb{R}^d)$, $\dot{T}_u v \in \dot{B}^{s+\sigma}_{p,r}$ and there exists C (depending only on $|s+\sigma|$) such that*

$$\|\dot{T}_u v\|_{\dot{B}^{s+\sigma}_{p,r}} \leqslant C\|u\|_{\dot{B}^\sigma_{\infty,r_1}}\|v\|_{\dot{B}^s_{p,r_2}}, \ \text{with } \frac{1}{r} = \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}.$$

(ii) Let $(s,\sigma) \in \mathbb{R}^2$ with $s + \sigma > 0$.
Let $(s+\sigma, p, r) \in \mathbb{R}^+ \times [1,\infty]^2$ satisfy $(*)$, $(p,p_1,p_2) \in [1,\infty]^3$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $(r,r_1,r_2) \in [1,\infty]^3$ satisfy $\frac{1}{r} = \min(1, \frac{1}{r_1} + \frac{1}{r_2})$.
Then for $u \in \dot{B}^s_{p_1,r_1}(\mathbb{R}^d)$ and $v \in \dot{B}^\sigma_{p_2,r_2}(\mathbb{R}^d)$, $\dot{R}(u,v) \in \dot{B}^{s+\sigma}_{p,r}(\mathbb{R}^d)$ and there exists C (depending only on $s+\sigma$) such that

$$\|\dot{R}(u,v)\|_{\dot{B}^{s+\sigma}_{p,r}} \leqslant C\|u\|_{\dot{B}^s_{p_1,r_1}}\|v\|_{\dot{B}^\sigma_{p_2,r_2}}.$$

Then we have the following estimates for the products:

(a) $L^\infty \cap \dot{B}^s_{p,r}(\mathbb{R}^d)$, $s > 0$ with (s,p,r) satisfying $(*)$ is an algebra:

$$\|uv\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} \leqslant C(\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} + \|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)}\|v\|_{L^\infty}).$$

(b) For $(s,\sigma)\in(-\frac{d}{2},\frac{d}{2})^2$ such that $s+\sigma>0$, we have

$$\|uv\|_{\dot B^{s+\sigma-\frac{d}{2}}_{2,1}(\mathbb{R}^d)}\leqslant C\|u\|_{\dot B^s_{2,2}(\mathbb{R}^d)}\|v\|_{\dot B^\sigma_{2,2}(\mathbb{R}^d)},$$

and in particular

$$\|uv\|_{\dot H^{s+\sigma-\frac{d}{2}}(\mathbb{R}^d)}\leqslant C\|u\|_{\dot H^s(\mathbb{R}^d)}\|v\|_{\dot H^\sigma(\mathbb{R}^d)}.$$

Proof. Proof of (i):
Since $\dot T_u v=\sum_{j\in\mathbb{Z}}\dot S_{j-1}u\dot\Delta_j v$ with $\widehat{\dot S_{j-1}u\dot\Delta_j}v$ compactly supported on the annulus $\{\xi\in\mathbb{R}^d\mid\frac{1}{3}2^j\leqslant|\xi|\leqslant\frac{10}{3}2^j\}$, we apply Theorem 2.1 and Bernstein's inequality to derive

$$\begin{aligned}\|\dot T_u v\|_{\dot B^s_{p,r}}&\leqslant C\big\|\big(2^{js}\|\dot S_{j-1}u\dot\Delta_j v\|_{L^p}\big)\big\|_{\ell^r}\leqslant C\big\|\big(2^{js}\|\dot S_{j-1}u\|_{L^\infty}\|\dot\Delta_j v\|_{L^p}\big)\big\|_{\ell^r}\\&\leqslant C\sup{}_{j\in\mathbb{Z}}\|\dot S_{j-1}u\|_{L^\infty}\big\|\big(2^{js}\|\dot\Delta_j v\|_{L^p}\big)\big\|_{\ell^r}\leqslant C\|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}}.\end{aligned}$$

We claim (**Exercise**) that if $\sigma<0$, then

$$\Big\|\big(2^{j\sigma}\|\dot S_j u\|_{L^p}\big)\Big\|_{\ell^r}\leqslant C\|u\|_{\dot B^\sigma_{p,r}},\tag{2.18}$$

and hence the same argument as above yields

$$\|\dot T_u v\|_{\dot B^{s+\sigma}_{p,r}}\leqslant C\big\|\big(2^{j\sigma}\|\dot S_{j-1}u\|_{L^\infty}\big)\big\|_{\ell^{r_1}}\big\|\big(2^{js}\|\dot\Delta_j v\|_{L^p}\big)\big\|_{\ell^{r_2}}=C\|u\|_{\dot B^\sigma_{\infty,r_1}}\|v\|_{\dot B^s_{p,r_2}}.$$

Proof of (ii):
Since $s+\sigma>0$ and $\dot R(u,v)=\sum_{j\in\mathbb{Z}}(\sum_{|j-k|\leqslant1}\dot\Delta_j u\dot\Delta_k v)$ with $\widehat{\sum_{|j-k|\leqslant1}\dot\Delta_j}u\dot\Delta_k v$ compactly supported on the ball of size $8\cdot2^j$, we derive from Theorem 2.1 that

$$\begin{aligned}&\|\dot R(u,v)\|_{\dot B^{s+\sigma}_{p,r}}\leqslant C\big\|\big(2^{j(s+\sigma)}\|\sum_{|j-k|\leqslant1}\dot\Delta_j u\dot\Delta_k v\|_{L^p}\big)\big\|_{\ell^r}\\&\leqslant C\big\|\big(2^{js}\|\dot\Delta_j u\|_{L^{p_1}}\big)\big\|_{\ell^{r_1}}\big\|\big(2^{j\sigma}\sum_{|j-k|\leqslant1}\|\dot\Delta_k v\|_{L^{p_2}}\big)\big\|_{\ell^{r_2}}\leqslant C\|u\|_{\dot B^s_{p_1,r_1}}\|v\|_{\dot B^\sigma_{p_2,r_2}}.\end{aligned}$$

Proof of (a):
We simply derive from (i) and (ii) that

$$\begin{aligned}\|uv\|_{\dot B^s_{p,r}}&\leqslant\|\dot T_u v\|_{\dot B^s_{p,r}}+\|\dot T_v u\|_{\dot B^s_{p,r}}+\|\dot R(u,v)\|_{\dot B^s_{p,r}}\\&\leqslant C\|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}}+C\|v\|_{L^\infty}\|u\|_{\dot B^s_{p,r}}+C\|u\|_{\dot B^0_{\infty,\infty}}\|v\|_{\dot B^s_{p,r}},\end{aligned}$$

which, together with the fact $\|u\|_{\dot{B}^0_{\infty,\infty}} = \sup_{j\in\mathbb{Z}}\|\dot{\Delta}_j u\|_{L^\infty} \leqslant C\|u\|_{L^\infty}$, implies the result.
Proof of (b):
Since $\dot{H}^s = \dot{B}^s_{2,2} \hookrightarrow \dot{B}^{s-\frac{d}{2}}_{\infty,2}$, we derive from (i) that

$$\|\dot{T}_u v\|_{\dot{B}^{s+\sigma-\frac{d}{2}}_{2,1}} \leqslant C\|u\|_{\dot{B}^{s-\frac{d}{2}}_{\infty,2}}\|v\|_{\dot{B}^\sigma_{2,2}} \leqslant C\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^\sigma},$$

and similarly we have the above inequality for $\dot{T}_v u$. By the embedding $\dot{B}^{s+\sigma}_{1,1} \hookrightarrow \dot{B}^{s+\sigma-\frac{d}{2}}_{2,1}$ we have from (ii) that

$$\|\dot{R}(u,v)\|_{\dot{B}^{s+\sigma-\frac{d}{2}}_{2,1}} \leqslant \|\dot{R}(u,v)\|_{\dot{B}^{s+\sigma}_{1,1}} \leqslant C\|u\|_{\dot{B}^s_{2,2}}\|v\|_{\dot{B}^\sigma_{2,2}}.$$

□

Corollary 2.1 (Hardy's Inequality)**.** *Let $s \in [0, \frac{d}{2})$, then there exists C such that*

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}}\,\mathrm{d}x \leqslant C\|f\|^2_{\dot{H}^s(\mathbb{R}^d)}.$$

Proof. **Exercise.** Notice that

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}}\,\mathrm{d}x \leqslant \sum_{|j-k|\leqslant 1} \|\dot{\Delta}_j(|f|^2)\|_{L^2}\|\dot{\Delta}_k(|x|^{-2s})\|_{L^2} \leqslant C\||f|^2\|_{\dot{B}^{2s-\frac{d}{2}}_{2,1}}\||x|^{-2s}\|_{\dot{B}^{\frac{d}{2}-2s}_{2,\infty}}.$$

□

Remark 2.5. *If $s+\sigma = 0$ and $r = 1$, then from the above proof we have*

$$\|\dot{R}(u,v)\|_{\dot{B}^0_{p,\infty}} \leqslant C\|u\|_{\dot{B}^s_{p_1,r_1}}\|v\|_{\dot{B}^{-s}_{p_2,r_2}}.$$

If $d \geqslant 3$ and $s = 1$, then we simply have the following Hardy's inequality

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2}\,\mathrm{d}x \leqslant C\|\nabla f\|^2_{L^2}.$$

[15.12.2021]

[17.12.2021]

2.2 Nonhomogeneous Besov spaces

2.2.1 Nonhomogeneous Besov spaces

In this subsection we give the definition of the nonhomogeneous Besov space. Most of the results in Subsection 2.1 still hold true in the nonhomogeneous framework, and the proofs are even simpler as we take the nonhomogeneous Littlewood-Paley decomposition (2.12):

$$u = \Delta_{-1}u + \sum_{j\geqslant 0}\Delta_j u, \quad u\in\mathcal{S}'(\mathbb{R}^d)$$

and we do not have to worry about the low frequency dyadic pieces $\dot{\Delta}_j u$, $j\leqslant -1$.

Definition 2.4. *Let $s\in\mathbb{R}$, $(p,r)\in[1,\infty]^2$. The nonhomogeneous Besov space $B^s_{p,r}$ consists of all tempered distributions $u\in\mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|u\|_{B^s_{p,r}} := \Big\|\big(2^{js}\|\Delta_j u\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r} < \infty.$$

Remark 2.6. *When $p=r=2$, then*

$$B^s_{2,2} = H^s := \{u\in\mathcal{S}' \mid \|u\|^2_{H^s} := \int_{\mathbb{R}^d}(1+|\xi|^2)^s|\hat{u}|^2\,\mathrm{d}\xi \ <\infty\}.$$

When $s\in(0,1)$, $p=r=\infty$, then

$$B^s_{\infty,\infty} = C^s := \{u\in C_b(\mathbb{R}^d) \mid [u]_s := \sup{}_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^s} <\infty\}.$$

We just list the properties that still hold true in the nonhomogeneous setting (proofs are left to the interested readers):

- The embedding property and interpolation property in Proposition 2.2 hold true in the nonhomogeneous setting.
- Theorem 2.1 holds in the nonhomogeneous setting except that we now have the facts that $B^s_{p,r}(\mathbb{R}^d)$ is a Banach space satisfying Fatou's property and $\mathcal{D}(\mathbb{R}^d)$ is dense in $B^s_{p,r}(\mathbb{R}^d)$ when $p<\infty$ and $r<\infty$.
- Theorem 2.2 holds in the nonhomogeneous setting, where the nonhomogeneous paraproduct is defined by

  $$T_u v = \sum_j S_{j-1}u\Delta_j v, \quad S_{j-1}u = \sum_{-1\leqslant k\leqslant j-2}\Delta_k u,$$

  and the nonhomgeneous remainder is defined by

  $$R(u,v) = \sum_{|k-j|\leqslant 1}\Delta_j u\Delta_k v.$$

2.2.2 Commutator estimates

Let us consider the transport equation

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = f, \quad t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{2.19}$$

where the unknown function $\rho = \rho(t,x) : \mathbb{R} \times \mathbb{R}^d \in \mathbb{R}$ is transported by the velocity vector field $v = v(t,x) : \mathbb{R} \times \mathbb{R}^d \in \mathbb{R}^d$ and $f = f(t,x) : \mathbb{R} \times \mathbb{R}^d \in \mathbb{R}$ is the source term.

At the first sight we may guess that if we would like to propagate the $C^s(\mathbb{R}^d)$-regularity of ρ_0, then we might need the terms $v \cdot \nabla \rho$, f to be in $C^s(\mathbb{R}^d)$, and thus ρ to be in $C^{s+1}(\mathbb{R}^d)$ which is not possible. Indeed, we may use the commutator estimates to control the commutator such as

$$[v \cdot \nabla, \partial^s]\rho = v \cdot \nabla \partial^s \rho - \partial^s(v \cdot \nabla \rho).$$

Roughly speaking, we would like to move one derivative ∂ from ρ to v. As we are working in the Besov setting, we are going to consider the commutator $[v \cdot \nabla, \Delta_j]\rho$.

We apply the operator Δ_j to it to arrive at the transport equation for $\Delta_j \rho$:

$$\begin{cases} \partial_t(\Delta_j \rho) + v \cdot \nabla(\Delta_j \rho) = \Delta_j f + [v \cdot \nabla, \Delta_j]\rho, \\ (\Delta_j \rho)|_{t=0} = \Delta_j \rho_0, \end{cases} \tag{2.20}$$

where $[v\cdot\nabla, \Delta_j]\rho$ denotes the commutator $v\cdot\nabla(\Delta_j\rho) - \Delta_j(v\cdot\nabla\rho)$. In order to transport the $B^s_{p,r}$-regularity of the unknown function ρ, we have to estimate the commutator as follows:

$$\left\| \left(2^{js} \|[v \cdot \nabla, \Delta_j]\rho\|_{L^p} \right)_{j \geq -1} \right\|_{\ell^r}.$$

(The proof of the homogeneous version is left to interested readers.)

Theorem 2.3 (Commutator estimates)**.** *Let $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$ satisfy*

$$\begin{aligned} &s > \max\{-\frac{d}{p}, -\frac{d}{p'}\} \textit{ with } \frac{1}{p'} = 1 - \frac{1}{p}, \\ &\textit{or } s > -1 + \max\{-\frac{d}{p}, -\frac{d}{p'}\} \textit{ if } \operatorname{div} v := \sum_{j=1}^d \partial_{x_j} v_j = 0. \end{aligned} \tag{2.21}$$

Let $v \in \mathcal{S}(\mathbb{R}^d;\mathbb{R}^d)$ be a smooth vector field.

Then there exists a constant C such that the following commutator estimate holds:

$$\left\| \left(2^{js} \|[v \cdot \nabla, \Delta_j]\rho\|_{L^p} \right)_{j \geq -1} \right\|_{\ell^r} \leq C \|\nabla v\|_{B^{\frac{d}{p}}_{p,\infty} \cap L^\infty} \|\rho\|_{B^s_{p,r}} \textit{ if } s < 1 + \frac{d}{p}. \tag{2.22}$$

For general $s>0$ or $s>-1$ when $\operatorname{div} v=0$*, then*

$$\Big\|\big(2^{js}\|[v\cdot\nabla,\Delta_j]\rho\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}\leqslant C\big(\|\nabla v\|_{L^\infty}\|\rho\|_{B^s_{p,r}}+\|\nabla v\|_{B^{s-1}_{p,r}}\|\nabla\rho\|_{L^\infty}\big). \quad (2.23)$$

[17.12.2021]

[22.12.2021]

Proof. We decompose v into the low frequency and high frequency parts:

$$v=\Delta_{-1}v+\tilde v,\quad \tilde v=\sum_{j\geqslant 0}\Delta_j v.$$

To warm up, we are going to show for the low frequency part that

$$\Big\|\big(2^{js}\|[\Delta_{-1}v\cdot\nabla,\Delta_j]\rho\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}\leqslant C\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{B^{s-1}_{p,r}}. \quad (2.24)$$

We have to keep in mind the supports of the Fourier transform, and the definitions of the operators

$$\Delta_{-1}=\chi(D)=(2\pi)^{-\frac d2}\check\chi *,$$
$$\Delta_j=\varphi_j(D)=\varphi(2^{-j}D)=(2\pi)^{-\frac d2}\check\varphi_j *=(2\pi)^{-\frac d2}2^{jd}\check\varphi(2^j\cdot)* \text{ for } j\geqslant 0.$$

- If $j=-1,0$, then the Fourier transforms of

  $$\Delta_{-1}v\cdot\nabla\Delta_j\rho \text{ and } \Delta_j(\Delta_{-1}v\cdot\nabla\rho)$$

  are compactly supported on the ball with radius smaller than 4, and

  $$[\Delta_{-1}v\cdot\nabla,\Delta_j]\rho=[\Delta_{-1}v\cdot\nabla,\Delta_j](\chi(4^{-1}D)\rho).$$

  If $j=-1$, then

  $$[\Delta_{-1}v\cdot\nabla,\Delta_{-1}]\rho=(2\pi)^{-\frac d2}\int_{\mathbb R^d}\check\chi(x-y)\big((\Delta_{-1}v)(x)-(\Delta_{-1}v)(y)\big)\cdot\nabla(\chi(4^{-1}D)\rho)(y)\,\mathrm dy$$
  $$=(2\pi)^{-\frac d2}\int_{\mathbb R^d}\check\chi(x-y)\int_0^1(x-y)\cdot\nabla\Delta_{-1}v(y+\tau(x-y))d\tau\cdot\chi(4^{-1}D)(\nabla\rho)(y)\,\mathrm dy\,,$$

  which can be estimated as

  $$\|[\Delta_{-1}v\cdot\nabla,\Delta_{-1}]\rho\|_{L^p}\leqslant\|z\check\chi(z)\|_{L^1}\|\nabla\Delta_{-1}v\|_{L^\infty}\|\chi(4^{-1}D)(\nabla\rho)\|_{L^p}\leqslant C\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{B^{s-1}_{p,\infty}}.$$

  We follow exactly the same lines to arrive at the estimates for $j=0$, by changing χ into φ above.

- For $j \geqslant 1$, the Fourier transform of the commutator

$$[\Delta_{-1}v \cdot \nabla, \Delta_j]\rho$$

is compactly supported on an annulus $2^j\mathcal{C}(\frac{1}{3}, \frac{10}{3})$, and hence

$$[\Delta_{-1}v \cdot \nabla, \Delta_j]\rho = \sum_{|j'-j|\leqslant 3} [\Delta_{-1}v \cdot \nabla, \Delta_j](\Delta_{j'}\rho).$$

We calculate as above:

$$\begin{aligned}
&[\Delta_{-1}v \cdot \nabla, \Delta_j]\rho \\
&= (2\pi)^{-\frac{d}{2}} \sum_{|j'-j|\leqslant 3} \int_{\mathbb{R}^d} \check{\varphi}_j(x-y)\big((\Delta_{-1}v)(x) - (\Delta_{-1}v)(y)\big) \cdot \nabla(\Delta_{j'}\rho)(y)\,\mathrm{d}y \\
&= (2\pi)^{-\frac{d}{2}} \sum_{|j'-j|\leqslant 3} \int_{\mathbb{R}^d} \check{\varphi}_j(x-y)\Big((x-y) \cdot \int_0^1 (\nabla\Delta_{-1}v)(y+\tau(x-y))d\tau\Big) \cdot (\Delta_{j'}\nabla\rho)(y)\,\mathrm{d}y\,.
\end{aligned}$$

Noticing

$$\check{\varphi}_j(x-y)\,(x-y) = 2^{jd}\check{\varphi}(2^j(x-y))\,(x-y) = 2^{-j}\Big(2^{jd}\big(z\check{\varphi}(z)\big)|_{z=2^j(x-y)}\Big),$$

we have

$$\|[\Delta_{-1}v \cdot \nabla, \Delta_j]\rho\|_{L^p} \leqslant \sum_{|j'-j|\leqslant 3} \|\nabla\Delta_{-1}v\|_{L^\infty} 2^{-j}\|z\check{\varphi}(z)\|_{L^1}\|\nabla\Delta_{j'}\rho\|_{L^p},$$

and hence

$$\begin{aligned}
\Big\|\big(2^{js}\|[\Delta_{-1}v \cdot \nabla, \Delta_j]\rho\|_{L^p}\big)_{j\geqslant 1}\Big\|_{\ell^r} &\leqslant C\|\nabla v\|_{L^\infty}\Big\|\big(\sum_{|j'-j|\leqslant 3} 2^{(j'-j)}2^{j'(s-1)}\|\Delta_{j'}\nabla\rho\|_{L^p}\big)_{j\geqslant 1}\Big\|_{\ell^r} \\
&\leqslant C\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{B^{s-1}_{p,r}}.
\end{aligned}$$

Now we consider the commutator

$$\begin{aligned}
&[\tilde{v} \cdot \nabla, \Delta_j]\rho = \sum_{k=1}^d \big(\tilde{v}^k\partial_k\Delta_j\rho - \Delta_j(\tilde{v}^k\partial_k\rho)\big) \\
&= \sum_{k=1}^d T_{\tilde{v}^k}\partial_k(\Delta_j\rho) + T_{\partial_k\Delta_j\rho}\tilde{v}^k + R(\tilde{v}^k, \partial_k\Delta_j\rho) - \Delta_j(T_{\tilde{v}^k}\partial_k\rho + T_{\partial_k\rho}\tilde{v}^k + R(\tilde{v}^k, \partial_k\rho)) \\
&= \sum_{k=1}^d [T_{\tilde{v}^k}, \Delta_j]\partial_k\rho + T_{\partial_k\Delta_j\rho}\tilde{v}^k - \Delta_j(T_{\partial_k\rho}\tilde{v}^k) + R(\tilde{v}^k, \partial_k\Delta_j\rho) - \Delta_j R(\tilde{v}^k, \partial_k\rho),
\end{aligned}$$

and we are going to estimate the terms one by one.

Step 1: Similarly as above, by virtue of the supports of the Fourier transform we have

$$\begin{aligned}
&[T_{\tilde{v}^k}, \Delta_j]\partial_k\rho = \sum_{j'\geqslant -1} [S_{j'-1}\tilde{v}^k, \Delta_j](\Delta_{j'}\partial_k\rho) \\
&= \sum_{j'\geqslant 1, |j-j'|\leqslant 1} S_{j'-1}\tilde{v}^k \Delta_j\Delta_{j'}\partial_k\rho - \Delta_j \sum_{j'\geqslant 1, |j-j'|\leqslant 3} S_{j'-1}\tilde{v}^k\Delta_{j'}\partial_k\rho \\
&= \sum_{j'\geqslant 1, |j-j'|\leqslant 3} [S_{j'-1}\tilde{v}^k, \Delta_j](\Delta_{j'}\partial_k\rho).
\end{aligned}$$

Correspondingly, we arrive at the following estimates

$$\begin{aligned}
&\Big\| \big(2^{js}\|[T_{\tilde{v}^k}, \Delta_j]\partial_k\rho\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r} \\
&\leqslant C\Big\|\big(2^{j(s-1)} \sum_{|j'-j|\leqslant 3} \|\nabla S_{j'-1}\tilde{v}\|_{L^\infty}\|\Delta_{j'}\nabla\rho\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r} \\
&\leqslant C\|\nabla v\|_{L^\infty}\Big\|\big(\sum_{|j'-j|\leqslant 3} 2^{(j'-j)(s-1)}2^{j'(s-1)}\|\Delta_{j'}\nabla\rho\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r} \\
&\leqslant C\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{B^{s-1}_{p,r}}.
\end{aligned}$$

Step 2: As

$$\begin{aligned}
\|T_{\partial_k\Delta_j\rho}\tilde{v}^k\|_{L^p} &= \Big\|\sum_{j'\geqslant j-3} S_{j'-1}\partial_k\Delta_j\rho\, \Delta_{j'}\tilde{v}^k\Big\|_{L^p} \\
&\leqslant C\|\partial_k\Delta_j\rho\|_{L^p}\sum_{j'\geqslant j-3}\|\Delta_{j'}\tilde{v}^k\|_{L^\infty} \\
&\leqslant C\|\Delta_j(\partial_k\rho)\|_{L^p}\sum_{j'\geqslant j-3} 2^{-j'}\|\Delta_{j'}(\nabla\tilde{v}^k)\|_{L^\infty} \\
&\leqslant C\|\Delta_j(\partial_k\rho)\|_{L^p}2^{-j}\|\nabla v\|_{L^\infty},
\end{aligned}$$

(2.24) follows for the term involving $T_{\partial_k\Delta_j\rho}\tilde{v}^k$.

Step 3: Now we come to

$$\big\|\Delta_j(T_{\partial_k\rho}\tilde{v}^k)\big\|_{L^p} = \Big\|\sum_{|j-j'|\leqslant 4}\Delta_j(S_{j'-1}\partial_k\rho\,\Delta_{j'}\tilde{v}^k)\Big\|_{L^p} \leqslant \sum_{|j-j'|\leqslant 4}\|S_{j'-1}\partial_k\rho\|_{L^\infty}\|\Delta_{j'}\tilde{v}^k\|_{L^p}.$$

Recall the characterisation (2.18) of $B^{s-1-\frac{d}{p}}_{\infty,r}$ when $s < 1 + \frac{d}{p}$, such that

$$\Big\|\big(2^{js}\|\Delta_j(T_{\partial_k\rho}\tilde{v}^k)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}$$

$$\leqslant C\Big\|\big(2^{j(s-1-\frac{d}{p})}\|S_{j-1}\nabla\rho\|_{L^\infty}\big)_{j\geqslant -1}\Big\|_{\ell^r}\Big\|\big(2^{j(1+\frac{d}{p})}\|\Delta_j\tilde{v}^k\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^\infty}$$
$$\leqslant C\|\nabla\rho\|_{B^{s-1-\frac{d}{p}}_{\infty,r}}\|\tilde{v}\|_{B^{1+\frac{d}{p}}_{p,\infty}} \leqslant C\|\nabla\rho\|_{B^{s-1}_{p,r}}\|\nabla v\|_{B^{\frac{d}{p}}_{p,\infty}}.$$

We can also simply estimate the above as (without any restriction on s):

$$\Big\|\big(2^{js}\|\Delta_j(T_{\partial_k\rho}\tilde{v}^k)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r} \leqslant C\|\nabla\rho\|_{L^\infty}\Big\|\big(2^{js}\|\Delta_j\tilde{v}^k\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}$$
$$\leqslant C\|\nabla\rho\|_{L^\infty}\|\nabla v\|_{B^{s-1}_{p,r}}.$$

Step 4: Next it is straightforward to estimate

$$\Big\|\big(2^{js}\|R(\tilde{v}^k,\partial_k\Delta_j\rho)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r} = \Big\|\big(2^{js}\|\sum_{|j-j'|\leqslant 2}\Delta_{j'}\tilde{v}^k\partial_k\Delta_j(\sum_{|j''-j'|\leqslant 1}\Delta_{j''}\rho)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}$$
$$\leqslant C\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{B^{s-1}_{p,r}}.$$

Step 5: Finally we can follow the idea in the proof of (ii) in Theorem 2.2 to control

$$\Big\|\big(2^{js}\|\Delta_j\sum_{k=1}^d R(\tilde{v}^k,\partial_k\rho)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}$$
$$= \Big\|\big(2^{js}\|\sum_{k=1}^d\Delta_j\partial_k R(\tilde{v}^k,\rho) - \Delta_j R(\operatorname{div}\tilde{v},\rho)\|_{L^p}\big)_{j\geqslant -1}\Big\|_{\ell^r}.$$

Indeed, in the general case where we do not know $\operatorname{div} v = 0$, we simply rewrite

$$2^{js}\|\Delta_j R(\tilde{v}^k,\partial_k\rho)\|_{L^p} \leqslant C\begin{cases} 2^{j(s+\frac{d}{p})}\|\Delta_j R(\tilde{v}^k,\partial_k\rho)\|_{L^{\frac{p}{2}}} & \text{if } s>-\frac{d}{p}\geqslant -\frac{d}{p'},\ \text{i.e. } p\geqslant 2,\\ 2^{j(s+\frac{d}{p'})}\|\Delta_j R(\tilde{v}^k,\partial_k\rho)\|_{L^1} & \text{if } s>-\frac{d}{p'}\geqslant -\frac{d}{p},\ \text{i.e. } p\leqslant 2,\end{cases}$$

then we make use of $\Delta_j R(\tilde{v}^k,\partial_k\rho) = \sum_{j'\geqslant j-4}\sum_{|j'-j''|\leqslant 1}\Delta_j\big((\Delta_{j'}\tilde{v}^k)(\Delta_{j''}\partial_k\rho)\big)$ to derive (2.22). If $\operatorname{div} v = 0$ such that $\operatorname{div}\tilde{v} = -\operatorname{div}\Delta_{-1}v$, then under the assumption $s > -1 + \max\{-\frac{d}{p}, -\frac{d}{p'}\}$ we still have (2.22). Under the assumption $s > 0$ or $s > -1$ if $\operatorname{div} v = 0$, the inequality (2.23) follows similarly. □

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[12.01.2022]

3 Applications to PDEs

3.1 Transport equation

In this section we will consider the transport equation (2.19):

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = f, & t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ \rho|_{t=0} = \rho_0, \end{cases} \tag{T}$$

where the function $\rho = \rho(t,x) \in \mathbb{R}$ is transported by the velocity vector field $v = v(t,x) \in \mathbb{R}^d$ and $f = f(t,x) \in \mathbb{R}$ is the source term.
We first have the following observations:

1. If the velocity vector field $v : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is smooth enough, e.g. $v \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^d))$, then the Cauchy-Lipschitz theorem implies the unique flow $\psi_t(\cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$, which is defined as the solution of initial value problem of the following ordinary differential equation (with $x \in \mathbb{R}^d$ viewed as a parameter):

$$\partial_t \psi_t(x) = v(t, \psi_t(x)), \quad \psi_t(x)|_{t=0} = x. \tag{3.25}$$

Thus the following function

$$\rho(t,x) = \rho_0(\psi_t^{-1}(x)) + \int_0^t f(t', \psi_{t'}(\psi_t^{-1}(x)))dt' \tag{3.26}$$

solves (at least formally) the Cauchy problem of the transport equation (T), since

$$\frac{d}{dt}(\rho(t,\psi_t(x))) = \frac{d}{dt}\Big(\rho_0(x) + \int_0^t f(t', \psi_{t'}(x))dt'\Big) = f(t, \psi_t(x)),$$
$$\rho(0,x) = \rho_0(x),$$

and by the definition of the flow $\psi_t(x)$:

$$\frac{d}{dt}(\rho(t,\psi_t(x))) = (\partial_t \rho)(t, \psi_t(x)) + v(t,\psi_t(x)) \cdot (\nabla \rho)(t, \psi_t(x)).$$

The resolution of the PDE (T) then reduces to the study of the ODE (3.25), and we will investigate the flow of a Lipschitz vector field in Subsection 3.1.1.

2. Compared to the Lagrangian point of view above, we would like to consider the PDE (T) directly: We apply Δ_j to (T) directly to derive the evolutionary equation for $\Delta_j\rho$ in (2.20):

$$\begin{cases} \partial_t(\Delta_j\rho) + v\cdot\nabla(\Delta_j\rho) = \Delta_j f + [v\cdot\nabla, \Delta_j]\rho, \\ (\Delta_j\rho)|_{t=0} = \Delta_j\rho_0. \end{cases} \tag{3.27}$$

We then use the commutator estimates in Theorem 2.3 for the commutator $[v\cdot\nabla, \Delta_j]\rho$ to derive the a priori estimates for the solutions of (T).

We will solve the PDE (T) in more general Besov spaces $B^s_{p,r}(\mathbb{R}^d)$ in Subsection 2.2.2.

3. Compared to the Lipschitz continuity assumption for the velocity vector field v above, DiPerna & Lions 1989 "Ordinary differential equations, transport theory and Sobolev spaces" have considered the velocity vector field of Sobolev-regulartiy:

$$v \in L^1([0,T]; W^{1,p'}(\mathbb{R}^d)), \quad \operatorname{div} v \in L^1([0,T]; L^\infty(\mathbb{R}^d)),$$

and showed the existence, uniqueness and stability results in the L^p-framework.

We explain a little why the L^∞-assumption on $\operatorname{div} v$ (instead of $\nabla v \in L^\infty(\mathbb{R}^d)$) is important for the $L^p(\mathbb{R}^d)$-theory of the transport equation from the following two points of view respectively:

- Assume that the flow ψ_t exists (not necessarily under DiPerna-Lions' weak assumption), then (**Exercise**)

$$\frac{d}{dt}(\det(\nabla\psi_t)(t,x)) = (\operatorname{div} v)(t,\psi_t(x))\cdot(\det(\nabla\psi_t)(x)), \tag{3.28}$$

which implies

$$\|\det(\nabla\psi_t)(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leqslant e^{\int_0^t \|\operatorname{div} v\|_{L^\infty(\mathbb{R}^d)}}.$$

In particular, if $\operatorname{div} v = 0$, then $\det(\nabla\psi_t) = 1$ and ψ_t is a measure-preserving flow.

If $\rho_0 \in L^p(\mathbb{R}^d)$, $p\in[1,\infty)$, then by virtue of (3.26),

$$\begin{aligned}\|\rho(t,\cdot)\|^p_{L^p(\mathbb{R}^d)} &\leqslant \int_{\mathbb{R}^d} |\rho_0(\psi_t^{-1}(x))|^p \,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^d} |f(t',\psi_{t'}(\psi_t^{-1}(x)))|^p \,\mathrm{d}x\, dt' \\ &= \int_{\mathbb{R}^d} |\rho_0(x)|^p |\det(\nabla\psi_t)| \,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^d} |f(t',x)|^p |\det(\nabla\psi_{t-t'})| \,\mathrm{d}x\ \mathrm{d}t',\end{aligned}$$

which is finite for finite time, if $\operatorname{div} v \in L^1_{\mathrm{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^d))$.

- For $p \in [1, \infty)$, we can simply test the equation (T) by $p\rho|\rho|^{p-2}$ and do integration by parts, such that

$$\frac{d}{dt}\int_{\mathbb{R}^d} |\rho|^p \, \mathrm{d}x = \int_{\mathbb{R}^d} (\operatorname{div} v)|\rho|^p \, \mathrm{d}x + \int_{\mathbb{R}^d} p\rho|\rho|^{p-2} f \, \mathrm{d}x$$
$$\leqslant \|\operatorname{div} v\|_{L^\infty} \|\rho\|_{L^p}^p + p\|\rho\|_{L^p}^{p-1} \|f\|_{L^p}$$

which implies

$$\frac{d}{dt}\|\rho\|_{L^p} \leqslant \frac{1}{p}\|\operatorname{div} v\|_{L^\infty}\|\rho\|_{L^p} + \|f\|_{L^p}.$$

By Gronwall's inequality we have the following a priori estimate

$$\|\rho(t)\|_{L^p(\mathbb{R}^d)} \leqslant e^{\frac{1}{p}\int_0^t \|\operatorname{div} v\|_{L^\infty(\mathbb{R}^d)}} \Big(\|\rho_0\|_{L^p(\mathbb{R}^d)} + \int_0^t \|f(t')\|_{L^p(\mathbb{R}^d)} \, \mathrm{d}t' \Big). \tag{3.29}$$

4. Finally, we consider a two dimensional model, where we denote the space variable $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ and take the velocity vector field

$$v(t, x, y) = \begin{pmatrix} y \\ 0 \end{pmatrix}. \tag{3.30}$$

We calculate straightforward

- $\nabla v = (\partial_{x_j} v^i)_{ij} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$;
- $\operatorname{div} v = \sum_{j=1}^d (\partial_{x_j} v^j) = 0$;
- The flow determined by (3.25) is $\psi_t(x, y) = \begin{pmatrix} x + yt \\ y \end{pmatrix}$.

For simplicity we take $f = 0$, and the Cauchy problem (T) reads as

$$\partial_t \rho + y\partial_x \rho = 0, \quad \rho|_{t=0} = \rho_0, \tag{3.31}$$

and the unique solution reads as

$$\rho(t, x, y) = \rho_0(\psi_t^{-1}(x, y)) = \rho_0(x - yt, y). \tag{3.32}$$

It is straightforward to check that

$$\|\rho(t)\|_{L^p(\mathbb{R}^2)}^p = \int_{\mathbb{R}^2} |\rho_0(x - yt, y)|^p \, \mathrm{d}x \, \mathrm{d}y = \|\rho_0\|_{L^p(\mathbb{R}^2)}^p, \quad \forall p \in [1, \infty);$$

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^2)} = \|\rho_0\|_{L^\infty(\mathbb{R}^2)},$$

while

$$\begin{aligned}
&\|\partial_x \rho(t)\|_{L^p(\mathbb{R}^2)} = \|\partial_x \rho_0\|_{L^p(\mathbb{R}^2)}, \quad \forall p \in [1,\infty], \\
&\|\partial_y \rho(t)\|_{L^p(\mathbb{R}^2)} = \|-t\partial_x \rho_0 + \partial_y \rho_0\|_{L^p(\mathbb{R}^2)} = O(|t|) \text{ if } \partial_x \rho_0 \neq 0,
\end{aligned}$$

and more generally, for any $k \in \mathbb{N}$,

$$\begin{aligned}
&\|\partial_x^k \rho(t)\|_{L^p(\mathbb{R}^2)} = \|\partial_x^k \rho_0\|_{L^p(\mathbb{R}^2)}, \quad \forall p \in [1,\infty], \\
&\|\partial_y^k \rho(t)\|_{L^p(\mathbb{R}^2)} = \|(-t\partial_x + \partial_y)^k \rho_0\|_{L^p(\mathbb{R}^2)} = O(|t|^k) \text{ if } \partial_x^k \rho_0 \neq 0.
\end{aligned}$$

The growth in the high-regularity-norm of the solution can also be illustrated by virtue of the Fourier transform. We take Fourier transform to the transport equation in (3.31) to derive

$$\partial_t \hat{\rho}(t,k,\eta) - k\partial_\eta \hat{\rho}(t,k,\eta) = 0, \quad \hat{\rho}(t,k,\eta)|_{t=0} = \hat{\rho}_0(k,\eta). \tag{3.33}$$

Thus there exists a unique solution

$$\hat{\rho}(t,k,\eta) = \hat{\rho}_0(k,\eta + kt).$$

If initially $\hat{\rho}_0$ is compactly supported near the point (k_0, η_0), then the solution $\hat{\rho}(t,k,\eta)$ is compactly supported near the point $(k_0, \eta_0 - k_0 t)$. That is, the "information" is transported to high frequency $|\eta_0 - k_0 t| = O(|t|)$ by the vector field (3.30). One can calculate straightforward the growth of the Sobolev norms

$$\begin{aligned}
\|\rho(t)\|^2_{\dot{H}^s(\mathbb{R}^d)} &\sim \int_{\mathbb{R}^d} (|k| + |\eta|)^{2s} |\hat{\rho}(t,k,\eta)|^2 dk d\eta \\
&= \int_{\mathbb{R}^d} (|k| + |\eta - kt|)^{2s} |\hat{\rho}_0(k,\eta)|^2 dk d\eta
\end{aligned}$$

which is of size $O(|t|^{2s})$ if $|\partial_x|^s \rho_0 \neq 0$.

3.1.1 Flow of a Lipschitz-continuous vector field

We first recall the standard Cauchy-Lipschitz theorem in the ODE theory.

Proposition 3.1 (Existence of the flow). *Let E be a Banach space, $\Omega \subset E$ an open set, $I \ni 0$ an open time interval and $X_0 \in \Omega$.*
Let $v : I \times \Omega \mapsto E$ be $L^1_{\mathrm{loc}}(I; \mathrm{Lip}(\Omega; E))$ in the following sense:

$$\int_K \sup_{\{(X_1(t), X_2(t)) \in \Omega^2 \,|\, X_1(t) \neq X_2(t)\}} \frac{\|v(t, X_1(t)) - v(t, X_2(t))\|_E}{\|X_1(t) - X_2(t)\|_E} dt < \infty,$$

for all compact set K in I.
Then there exists an open time interval $J \ni 0$ (with $J \subset I$) such that the equation

$$X(t) = X_0 + \int_0^t v(t', X(t'))\,\mathrm{d}t', \tag{ODE}$$

i.e.

$$\frac{d}{dt}X(t) = v(t, X(t)), \text{ with } X(0) = X_0$$

has a unique continuous solution $X = X(t) : J \mapsto \Omega$.

Proof. The theorem follows from the Picard iteration scheme:

$$X_{k+1}(t) = X_0 + \int_0^t v(t', X_k(t'))\,\mathrm{d}t', \quad \forall k \geqslant 0.$$

Indeed, since

$$\begin{aligned}\|X_{k+1} - X_k\|_E &\leqslant \int_0^t \|v(t', X_k(t')) - v(t', X_{k-1}(t'))\|_E\,\mathrm{d}t' \\ &\leqslant \int_0^t \|X_k(t') - X_{k-1}(t')\|_E \gamma(t')\,\mathrm{d}t',\end{aligned}$$

where the function $\gamma = \gamma(t)$ characterizes the Lipschitz dependence of the function $v = v(t, \cdot)$:

$$\gamma(t) := \sup_{\{(X_1(t), X_2(t)) \in \Omega^2 \,|\, X_1(t) \neq X_2(t)\}} \frac{\|v(t, X_1(t)) - v(t, X_2(t))\|_E}{\|X_1(t) - X_2(t)\|_E},$$

then there exists some time interval $J \ni 0$ such that the sequence $X_0 + \sum_{k\geqslant 0}(X_{k+1}(t) - X_k(t))$ converges to $X(t)$ in $\Omega \subset E$ uniformly on J by virtue of

$$\sup_{t\in J} \|X_{k+1}(t) - X_k(t)\|_E \leqslant \sup_{t\in J} \|X_k(t) - X_{k-1}(t)\|_E \int_J \gamma(t')\,\mathrm{d}t'$$

and $\gamma \in L^1_{\mathrm{loc}}(I)$. Hence $X(t) = X_0 + \sum_{k\geqslant 0}(X_{k+1} - X_k) \in C(J; \Omega)$ is the solution of (ODE) and is unique by a similar argument as above. □

[12.01.2022]

[14.01.2022]

Proposition 3.2 (Generated flow)**.** *Assume the hypotheses in Theorem 3.1 with* $\Omega = E = \mathbb{R}^d$*,* $x \in \mathbb{R}^d$ *and* $v \in L^1_{\mathrm{loc}}(I; \mathrm{Lip}(\mathbb{R}^d; \mathbb{R}^d))$*. Then the flow* $\psi_t = \psi(t,\cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ *of the vector field* v*:*

$$\psi_t(x) = x + \int_0^t v(t', \psi_{t'}(x))\,\mathrm{d}t' \tag{3.34}$$

is a homeomorphism on $\mathbb{R}^d$ *on the entire time interval* I*. Furthermore,* $\psi(t,x) \in C(I; C_b(\mathbb{R}^d;\mathbb{R}^d))$ *satisfies*

$$\begin{aligned} &\|\nabla\psi_t^{\pm 1}\|_{L^\infty} \leqslant e^{d|\int_0^t \|\nabla v\|_{L^\infty}|},\\ &\|\nabla\psi_t^{\pm 1} - \mathrm{Id}\,\|_{L^\infty} \leqslant e^{d|\int_0^t \|\nabla v\|_{L^\infty}|} - 1.\\ &\|\nabla^2\psi_t^{\pm 1}\|_{L^\infty} \leqslant e^{d|\int_0^t \|\nabla v\|_{L^\infty}|}\Big|\int_0^t d^2\|\nabla^2 v\|_{L^\infty} e^{2d|\int_0^{t'} \|\nabla v\|_{L^\infty}|}\,\mathrm{d}t'\Big|, \end{aligned} \tag{3.35}$$

where in the last inequality above we assume furthermore $\nabla v \in L^1_{\mathrm{loc}}(I; \mathrm{Lip}(\mathbb{R}^d; \mathbb{R}^{d\times d}))$*.*

Proof. By Proposition 3.1, for any fixed $t' \in \mathbb{R}$, $x \in \mathbb{R}^d$, there exists a unique solution $X(\cdot, t', x) : \mathbb{R} \mapsto \mathbb{R}^d$ solves the following initial value problem (with t', x viewed as parameters)

$$\frac{d}{dt}X(t,t',x) = v(t, X(t,t',x)), \quad X(t',t',x) = x.$$

Indeed, we first have a local-in-time existence and uniqueness results (with $\int_J \|v(t',\cdot)\|_{\mathrm{Lip}}\,\mathrm{d}t' \leqslant \frac{1}{2}$), which can be extended globally in time by the unique continuation argument. By the uniqueness results we have

$$X(t, t'', X(t'', t', x)) = X(t, t', x).$$

We define

$$\psi_t(x) = X(t, 0, x), \quad \psi_t^{-1}(x) = X(0, t, x),$$

such that

$$\psi_t \circ \psi_t^{-1} = \mathrm{Id}\,.$$

Without loss of generality we assume $t \geqslant 0$. Differentiate (3.34) with respect to x yields

$$\partial_{x_j}(\psi_t)^k = \delta_{j,k} + \int_0^t \sum_{l=1}^d \big((\partial_l v^k)(t', \psi_{t'})\big)\big(\partial_{x_j}(\psi_{t'})^l\big)\,\mathrm{d}t'.$$

Then the first inequality in (3.35) for ψ_t follows from Gronwall's lemma. Correspondingly we derive the second inequality in (3.35) for ψ_t as follows:

$$\|\nabla\psi_t - \mathrm{Id}\,\|_{L^\infty} \leqslant \int_0^t d\|\nabla v\|_{L^\infty} e^{d\int_0^{t'}\|\nabla v\|_{L^\infty}}\,\mathrm{d}t' = \int_0^t \mathrm{d}e^{d\int_0^t\|\nabla v\|_{L^\infty}} = e^{d\int_0^t\|\nabla v\|_{L^\infty}} - 1.$$

We can further differentiate with respect to x to derive

$$\begin{aligned}\partial_{jn}(\psi_t)^k = &\int_0^t \sum_{l,m=1}^d \big((\partial_{lm}v^k)(t',\psi_{t'})\big)\big(\partial_{x_j}(\psi_{t'})^l\big)\big(\partial_{x_n}(\psi_{t'})^m\big)\,\mathrm{d}t' \\ &+ \int_0^t \sum_{l=1}^d \big((\partial_l v^k)(t',\psi_{t'})\big)\big(\partial_{jn}(\psi_{t'})^l\big)\,\mathrm{d}t'.\end{aligned}$$

By Gronwall's lemma and the inequality (3.35) we derive

$$\|\nabla^2\psi_t\|_{L^\infty} \leqslant e^{d\int_0^t\|\nabla v\|_{L^\infty}} \int_0^t d^2\|\nabla^2 v\|_{L^\infty} e^{2d\int_0^{t'}\|\nabla v\|_{L^\infty}}\,dt'.$$

The inequalities for ψ_t^{-1} follow similarly (**Exercise**). □

3.1.2 The L^p-framework

Theorem 3.1 (Solvability of the transport equation in L^p)**.** *Assume the initial data $\rho_0 \in L^p(\mathbb{R}^d)$, $p \in [1,\infty]$, the source term $f \in L^1([0,T];L^p(\mathbb{R}^d))$, and the vector field $v \in L^1([0,T];W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$.*
Then the equation (T) *has a unique solution $\rho \in L^\infty([0,T];L^p(\mathbb{R}^d))$ in the distribution sense that for any $\phi \in C_c^\infty([0,T)\times\mathbb{R}^d)$ the following equality holds true:*

$$\begin{aligned}&-\int_0^T\int_{\mathbb{R}^d}\rho\partial_t\phi\,\mathrm{d}t\,\mathrm{d}x - \int_{\mathbb{R}^d}\rho_0\phi(0,x)\,\mathrm{d}x - \int_0^T\int_{\mathbb{R}^d}\mathrm{div}\,(v\phi)\rho\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_0^T\int_{\mathbb{R}^d} f\phi\,\mathrm{d}t\,\mathrm{d}x\,.\end{aligned} \tag{3.36}$$

Furthermore for a.e. $t \in [0,T]$, the estimate (3.29) *holds*

$$\|\rho(t)\|_{L^p(\mathbb{R}^d)} \leqslant e^{\frac{1}{p}\int_0^t\|\mathrm{div}\,v\|_{L^\infty(\mathbb{R}^d)}}\Big(\|\rho_0\|_{L^p(\mathbb{R}^d)} + \int_0^t\|f(t')\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}t'\Big), \tag{3.37}$$

and if $p < \infty$ then $\rho \in C([0,T];L^p)$.

Remark 3.1. *If the solution $\rho \in C^1([0,T] \times \mathbb{R}^d)$ of* (T) *is smooth enough, then we can simply test the equation by $\phi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ to derive*

$$\int_{\mathbb{R}^d} \partial_t \rho \phi \, \mathrm{d}x + \int_{\mathbb{R}^d} v \cdot \nabla \rho \phi \, \mathrm{d}x = \int_{\mathbb{R}} f \phi \, \mathrm{d}x \,.$$

As $\phi(T,x) = 0$, we take the time integration of the first integral on the left hand side from 0 *to* T *to derive*

$$-\int_0^T \int_{\mathbb{R}^d} \rho \partial_t \phi \, \mathrm{d}t \; \mathrm{d}x - \int_{\mathbb{R}^d} \rho_0 \phi(0,x) \, \mathrm{d}x \,.$$

We take integration by parts to rewrite the second integral on the left hand side as

$$-\int_{\mathbb{R}^d} \mathrm{div}\,(v\phi) \rho \, \mathrm{d}x \,.$$

And the formular (3.36) *follows.*
We observe then that under the assumptions of the theorem, all the integrals in (3.36) *are well-defined if $\rho \in L^\infty([0,T]; L^p(\mathbb{R}^d))$. However, provided with $\rho \in L^\infty([0,T]; L^p(\mathbb{R}^d))$ and $v \in L^1([0,T]; W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d))$, we may not define the product $v \cdot \nabla \rho$ in the transport equation* (T)*, while we can understand the term $v \cdot \nabla \rho$ in the following sense*

$$\mathrm{div}\,(\rho v) - (\mathrm{div}\, v)\rho \in L^1([0,T]; W^{-1,p}(\mathbb{R}^d)),$$

which is indeed the distributional sense stated in (3.36) *since*

$$\begin{aligned} (\mathrm{div}\,(\rho v) - (\mathrm{div}\, v)\rho, \phi)_{\mathcal{D}',\mathcal{D}} &= (-\rho v, \nabla \phi)_{\mathcal{D}',\mathcal{D}} - ((\mathrm{div}\, v)\rho, \phi)_{\mathcal{D}',\mathcal{D}} \\ &= -\int_0^T \int_{\mathbb{R}^d} \Big(\rho v \cdot \nabla \phi + (\mathrm{div}\, v) \rho \phi \Big) \, \mathrm{d}t \; \mathrm{d}x \\ &= -\int_0^T \int_{\mathbb{R}^d} \Big(\rho \mathrm{div}\,(v\phi) \Big) \, \mathrm{d}t \; \mathrm{d}x \,. \end{aligned}$$

Proof. We are going to follow the standard procedure to solve a PDE in four steps:

- Step 1 Establish the a priori estimates for smooth enough solutions;
- Step 2 Construct a sequence of approximated solutions, by the regularisation argument;
- Step 3 Show the convergence of the approximate solution sequence, by the compactness argument;

- Step 4 Verify the properties of the limit solution, by regularisation, approximation or duality arguments.

Step 1 A priori estimate
If $\rho \in (W^{1,p} \cap C_b^1)(\mathbb{R} \times \mathbb{R}^d)$ is a smooth enough solution of (T), then a priori estimate (3.29) holds true.
Step 2 Approximate solution sequence
Let η be a mollifier, such that $\eta \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \eta = 1$, $\eta \geqslant 0$, $\mathrm{Supp}\,(\eta) \subset B_1(0)$. Let $\eta_\varepsilon = \varepsilon^{-d}\eta(\varepsilon^{-1}\cdot)$. We regularise the data as follows [2]:

$$\begin{aligned}
&\rho_{0,\varepsilon} := \eta_\varepsilon * \rho_0 \in W^{\infty,p}(\mathbb{R}^d),\\
&v_\varepsilon := \eta_\varepsilon * v \in L^1([0,T]; C_b^\infty(\mathbb{R}^d;\mathbb{R}^d)),\\
&f_\varepsilon := \eta_\varepsilon * f \in L^1([0,T]; W^{\infty,p}(\mathbb{R}^d)).
\end{aligned}$$

Then the regularised system

$$\begin{cases} \partial_t \rho + v_\varepsilon \cdot \nabla \rho = f_\varepsilon, \quad t \in \mathbb{R},\ x \in \mathbb{R}^d, \\ \rho|_{t=0} = \rho_{0,\varepsilon}, \end{cases} \tag{3.38}$$

has a unique solution

$$\rho_\varepsilon(t,x) = \rho_{0,\varepsilon}(\psi_{t,\varepsilon}^{-1}(x)) + \int_0^t f_\varepsilon(t', \psi_{t',\varepsilon}(\psi_{t,\varepsilon}^{-1}(x)))dt' \in (W^{1,p} \cap C_b^1)([0,T] \times \mathbb{R}^d),$$

which satisfies the above a priori estimates independently of ε:

$$\|\rho_\varepsilon(t)\|_{L^p(\mathbb{R}^d)} \leqslant e^{\frac{1}{p}\int_0^t \|\mathrm{div}\, v\|_{L^\infty(\mathbb{R}^d)}} \Big(\|\rho_0\|_{L^p(\mathbb{R}^d)} + \int_0^t \|f(t')\|_{L^p(\mathbb{R}^d)}\, \mathrm{d}t' \Big). \tag{3.39}$$

Obviously the integral equality (3.36) holds true:

$$\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \partial_t \phi \,\mathrm{d}t\ \mathrm{d}x - \int_{\mathbb{R}^d} \rho_{0,\varepsilon}\phi(0,x)\,\mathrm{d}x - \int_0^T \int_{\mathbb{R}^d} \mathrm{div}\,(v_\varepsilon \phi)\rho_\varepsilon \,\mathrm{d}t\ \mathrm{d}x \\
&= \int_0^T \int_{\mathbb{R}^d} f_\varepsilon \phi \,\mathrm{d}t\ \mathrm{d}x\,.
\end{aligned}$$

[14.01.2022]

[19.01.2022]

Step 3 Convergence

[2]This regularisation $\eta_\varepsilon *$ can be easily compared with the application of the low frequency cut-off operator $S_j = 2^{jd}\check{\chi}(2^j *)*$ with $\varepsilon^{-1} \sim 2^j$. Recall the convergence results in Remark 2.1.

If $p \neq 1$, then as the approximated solution sequence is uniformly bounded in the space $L^\infty([0,T];L^p(\mathbb{R}^d))$, there exists a subsequence which converges weakly to the limit $\rho \in L^\infty([0,T];L^p(\mathbb{R}^d))$. This implies immediately for any $\phi \in C_c^\infty([0,T) \times \mathbb{R}^d)$,

$$-\int_0^T \int_{\mathbb{R}^d} \rho_\varepsilon \partial_t \phi \,\mathrm{d}t\ \mathrm{d}x \to -\int_0^T \int_{\mathbb{R}^d} \rho \partial_t \phi \,\mathrm{d}t\ \mathrm{d}x\,.$$

Notice the following convergence facts (up to a subsequence)

$$\rho_{0,\varepsilon} \underset{\text{if } p<\infty}{\overset{L^p}{\longrightarrow}} \rho_0, \quad \text{or} \quad \rho_{0,\varepsilon} \overset{*L^\infty}{\rightharpoonup} \rho_0;$$
$$f_\varepsilon \underset{\text{if } p<\infty}{\overset{L^1([0,T];L^p)}{\longrightarrow}} f, \quad \text{or} \quad f_\varepsilon \overset{*L^1([0,T];L^\infty)}{\rightharpoonup} f,$$

such that for any fixed $\phi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ the following convergence results hold:

$$\int_{\mathbb{R}^d} \rho_{0,\varepsilon} \phi(0,x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} \rho_0 \phi(0,x)\,\mathrm{d}x\,,$$
$$\int_0^T \int_{\mathbb{R}^d} f_\varepsilon \phi \,\mathrm{d}t\ \mathrm{d}x \to \int_0^T \int_{\mathbb{R}^d} f \phi \,\mathrm{d}t\ \mathrm{d}x\,.$$

Now we consider the term

$$\mathrm{div}\,(v_\varepsilon \phi) = (\mathrm{div}\,v)_\varepsilon \phi + v_\varepsilon \cdot \nabla \phi.$$

For a general function $g \in L^q(\mathbb{R}^d)$, we write

$$(\eta_\varepsilon * g)\phi = \eta_\varepsilon * (g\phi) + [\eta_\varepsilon *, \phi] g,$$

where the commutator reads more precisely as

$$([\eta_\varepsilon *, \phi] g)(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)(\phi(y) - \phi(x)) g(y)\,\mathrm{d}y\,.$$

We can follow exactly the idea of the proof of (2.24) to arrive at the following commutator estimate (**Exercise**):

$$\|[\eta_\varepsilon *, \phi] g\|_{L^q(\mathbb{R}^d)} \leqslant \varepsilon \|\nabla \phi\|_{L^\infty(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}, \quad \forall q \in [1,\infty].$$

Noticing $\mathrm{div}\,(v\phi) \in L^1([0,T];L^q(\mathbb{R}^d))$ as ϕ is compactly supported and $v \in L^1([0.T];W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$, we can then conclude

$$\mathrm{div}\,(v_\varepsilon \phi) - \mathrm{div}\,(v\phi)$$

$$
\begin{aligned}
&= \operatorname{div}(v_\varepsilon\phi) - \eta_\varepsilon * \operatorname{div}(v\phi) + \eta_\varepsilon * \operatorname{div}(v\phi) - \operatorname{div}(v\phi) \\
&= -[\eta_\varepsilon *, \phi](\operatorname{div} v) - [\eta_\varepsilon, \nabla\phi]\cdot v + \eta_\varepsilon * \operatorname{div}(v\phi) - \operatorname{div}(v\phi) \\
&\to 0 \quad \text{strongly in } L^1([0,T]; L^q(\mathbb{R}^d)), \quad \forall q \in [1,\infty).
\end{aligned}
$$

Therefore the following convergence result holds:

$$
\int_0^T \int_{\mathbb{R}^d} \operatorname{div}(v_\varepsilon\phi)\rho_\varepsilon \,\mathrm{d}t\ \mathrm{d}x \to \int_0^T \int_{\mathbb{R}^d} \operatorname{div}(v\phi)\rho \,\mathrm{d}t\ \mathrm{d}x\,,
$$

and $\rho \in L^\infty([0,T]; L^p(\mathbb{R}^d))$, $p \in (1,\infty]$ is a solution of (T) in the sense of (3.36). Furthermore, by virtue of the uniform estimates (3.39), the limit also satisfies the estimate (3.29).
If $p = 1$, then $\{\rho_\varepsilon\}$ is also weakly relatively compact in $L^\infty([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$, such that there exists a subsequence which converges weakly in it. Thus the limit $\rho \in L^\infty([0,T]; L^1(\mathbb{R}^d))$ also solves (T) in the sense of (3.36), provided one notices that there exists a function $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ taking value 1 on the support of ϕ, such that

$\rho_\varepsilon \operatorname{div}(v_\varepsilon\phi) = \rho_\varepsilon \operatorname{div}(v_\varepsilon\phi)\psi$,

and $\operatorname{div}(v_\varepsilon\phi) \xrightarrow{\text{a.e.}} \operatorname{div}(v\phi)$, $\quad \operatorname{div}(v_\varepsilon\phi)$ uniformly bounded in $L^1([0,T]; L^\infty(\mathbb{R}^d))$,

such that $\rho_\varepsilon \operatorname{div}(v_\varepsilon\phi) \overset{L^1([0,T];L^1_{\text{loc}}(\mathbb{R}^d))}{\rightharpoonup} \rho \operatorname{div}(v\phi)$.

In order to show the weak relative compactness of the sequence $\{\rho_\varepsilon\}$ in $L^\infty([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$, we would like to use (the a priori estimates for) the equation itself [3]: We take the regularised initial data $\rho_{0,n} := S_n\rho_0 \in W^{k,q}(\mathbb{R}^d)$ and the regularised force term $f_n := S_n f \in L^1([0,T]; W^{k,q}(\mathbb{R}^d))$, $\forall k \in \mathbb{N}, q \in [1,\infty]$, and solve the regularised system (3.38) to arrive at the unique solution $\rho_{n,\varepsilon}$. For any fixed n, for any fixed $q \in (1,\infty)$, by the above argument we have the uniform (independent of ε, but dependent on n, q) bounds on $\|\rho_{n,\varepsilon}\|_{L^\infty([0,T];L^q(\mathbb{R}^d))}$ which implies the weak compactness of $\{\rho_{n,\varepsilon}\}$ in $L^\infty([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$. On the other side, by the estimates (3.39) we have the following uniform (independent of ε) estimates:

$$
\begin{aligned}
&\|\rho_\varepsilon - \rho_{n,\varepsilon}\|_{L^\infty([0,T];L^1)} \\
&\leqslant e^{\int_0^T \|\operatorname{div} v\|_{L^\infty}} \left(\|\rho_{0,\varepsilon} - \rho_{0,n,\varepsilon}\|_{L^1(\mathbb{R}^d)} + \int_0^T \|f_\varepsilon - f_{n,\varepsilon}\|_{L^1(\mathbb{R}^d)} \,\mathrm{d}t \right) \\
&\leqslant e^{\int_0^T \|\operatorname{div} v\|_{L^\infty}} \left(\|\rho_0 - \rho_{0,n}\|_{L^1(\mathbb{R}^d)} + \int_0^T \|f - f_n\|_{L^1(\mathbb{R}^d)} \,\mathrm{d}t \right).
\end{aligned}
\tag{3.40}
$$

[3] Indeed the compactness comes from the time direction.

Step 4 Properties
Given the solution $\rho \in L^\infty([0,T]; L^p(\mathbb{R}^d))$, we consider its regularisation

$$\bar\rho_\varepsilon := \rho * \eta_\varepsilon,$$

which solves (not (3.38))

$$\begin{cases} \partial_t \bar\rho_\varepsilon + v \cdot \nabla \bar\rho_\varepsilon = f_\varepsilon + [v\cdot\nabla, \eta_\varepsilon *]\rho, \quad t \in [0,T],\ x \in \mathbb{R}^d, \\ \bar\rho_\varepsilon|_{t=0} = \rho_{0,\varepsilon}. \end{cases} \tag{3.41}$$

By a similar argument as above, we have the following convergence results for the commutator (**Exercise**):

$$[v\cdot\nabla, \eta_\varepsilon *]\rho \to 0 \text{ in } L^1([0,T]; L^p(\mathbb{R}^d)).$$

By use of this fact, we can deduce the renormalised equation for ρ:

$$\partial_t(b(\rho)) + v\cdot\nabla(b(\rho)) = b'(\rho) f, \tag{3.42}$$

for any function $b \in C^1_b(\mathbb{R})$.
If $p < \infty$, we may take $b(t) = (\min\{|t|, M\})^p$ for some fixed M (by use of an approximation argument), a smooth cut-off function ϕ_R with value 1 on the ball $B_R(0)$ while 0 outside the ball $B_{2R}(0)$, and test the renormalised equation (3.42) by ϕ_R. Finally letting $R \to \infty$, $M \to \infty$, for $p \in [1,\infty)$, the limit ρ satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^d} |\rho|^p \,\mathrm{d}x = \int_{\mathbb{R}^d} (\operatorname{div} v)|\rho|^p \,\mathrm{d}x + \int_{\mathbb{R}^d} p\rho|\rho|^{p-2} f \,\mathrm{d}x \ \text{ a.e. } [0,T], \tag{3.43}$$

which implies $\|\rho\|_{L^p(\mathbb{R}^d)} \in C([0,T])$, the uniqueness result and the uniform bound (3.37). Hence $\rho \in C([0,T]; L^p(\mathbb{R}^d))$ for $p \in (1,\infty)$ follows immediately.

[19.01.2022]

[26.01.2022]

The continuity for the case $p = 1$ needs a little more work: We take the solution $\rho_n \in C([0,T]; L^q(\mathbb{R}^d))$, $\forall q \in (1,\infty)$ of (T) with the data $\rho_{0,n} = S_n\rho_0$ and $f_n = S_n f$, and derive from (3.43) that (similar as (3.40))

$$\rho_n \overset{L^\infty([0,T];L^1(\mathbb{R}^d))}{\longrightarrow} \rho,$$

which implies $\rho \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$. In order to show $\rho \in C([0,T]; L^1(\mathbb{R}^d))$, one can simply test (3.42) with $b(t) = \min\{|t|, M\}$ by $1 - \phi_R$ to show that

$$\left\| \int_{\mathbb{R}^d} b(\rho)(1-\phi_R)\,\mathrm{d}x \right\|_{L^\infty([0,T])} \to 0 \text{ as } R \to \infty.$$

Thus $\rho \in C([0,T];L^1(\mathbb{R}^d))$.
The uniqueness result for $p = \infty$ follows from a duality argument, which we states here (see P. 519 the proof of Theorem II.2 in DiPerna-Lions' article): It suffices to show for $\rho_0 = 0$ and $f = 0$, the solution $\rho \in L^\infty([0,T] \times \mathbb{R}^d)$ satisfies

$$\int_0^T \int_{\mathbb{R}^d} \rho\phi \, \mathrm{d}t \; \mathrm{d}x \; = 0, \quad \forall \phi \in C_c^\infty((0,T) \times \mathbb{R}^d). \tag{3.44}$$

Recalling the weak formulation (3.45) of the transport equation for this solution $\rho \in L^\infty([0,T] \times \mathbb{R}^d)$:

$$\int_0^T \int_{\mathbb{R}^d} \rho\Big(\partial_t \Phi + \mathrm{div}\,(v\Phi)\Big) \, \mathrm{d}t \; \mathrm{d}x \; = 0, \quad \forall \Phi \in C_c^\infty([0,T) \times \mathbb{R}^d).$$

This leads us to consider the solution $\Phi \in L^\infty([0,T];L^1 \cap L^\infty)$ of the backward problem

$$\partial_t \Phi + \mathrm{div}\,(v\Phi) = \phi \text{ in } (0,T) \times \mathbb{R}^d, \quad \Phi|_{t=T} = 0.$$

We are going to use again the regularisation arguments to show $\rho = 0$, that is, the equality (3.44). Recalling the regularised problem (3.41) with the solution $\bar\rho_\varepsilon$, we consider the regularised backward problem

$$\partial_t \bar\Phi_\varepsilon + v \cdot \bar\nabla \Phi_\varepsilon + (\mathrm{div}\, v)\bar\Phi_\varepsilon = \phi_\varepsilon + [v \cdot \nabla, \eta_\varepsilon *]\Phi + [\mathrm{div}\, v, \eta_\varepsilon *]\Phi \text{ in } (0,T) \times \mathbb{R}^d, \quad \bar\Phi_\varepsilon|_{t=T} = 0,$$

with the commutators on the righthand side converging to 0 in $L^1([0,T];L^1(\mathbb{R}^d))$.
We then test the equation (3.41) by $\bar\Phi_\varepsilon \phi_R$ to derive (similar as (3.36))

$$-\int_0^T \int_{\mathbb{R}^d} \bar\rho_\varepsilon \Big(\partial_t(\bar\Phi_\varepsilon \phi_R) + \mathrm{div}\,(v\bar\Phi_\varepsilon \phi_R)\Big) \, \mathrm{d}t \; \mathrm{d}x \; = \int_0^T \int_{\mathbb{R}^d} ([v \cdot \nabla, \eta_\varepsilon *]\rho)(\bar\Phi_\varepsilon \phi_R) \, \mathrm{d}t \; \mathrm{d}x \,,$$

which implies (letting $\varepsilon \to 0$)

$$\int_0^T \int_{\mathbb{R}^d} \rho\phi = -\int_0^T \int_{\mathbb{R}^d} \rho\Phi v \cdot \nabla \phi_R \, \mathrm{d}t \; \mathrm{d}x \,.$$

We conclude the uniqueness result for $p = \infty$. □

Remark 3.2. *As the flow $\psi_t(x) \in C_b([0,T];W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$ generated by the velocity vector field $v \in L^1([0,T];W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$ exists, we can show immediately the existence of a solution* (3.26) *to* (T) *in the distribution sense instead of Step 1-Step 3 above, since we can verify straightforward that the function defined by* (3.26) *belongs to $L^\infty([0,T];L^p(\mathbb{R}^d))$ and satisfies* (3.36) *(**Exercise**). By Theorem 3.1,* (3.26) *is indeed the unique solution of* (T).
The DiPerna-Lions' method we followed here can indeed show more general results (when the generated flow may not necessarily exist): If $\rho_0 \in$

$L^p(\mathbb{R}^d)$, $p \in [1,\infty]$, $v \in L^1([0,T]; W^{1,p'}(\mathbb{R}^d;\mathbb{R}^d))$ *(which can be weakened to* $v \in L^1([0,T]; W^{1,p'}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d))$ *with some additional restrictions at infinity),* $\mathrm{div}\, v \in L^1([0,T]; L^\infty(\mathbb{R}^d))$, $f \in L^1([0,T]; L^p(\mathbb{R}^d))$, *then there exists a unique solution* $\rho \in L^\infty([0,T]; L^p(\mathbb{R}^d))$ *of* (T) *in the distribution sense* (3.36).
Notice that under the DiPerna-Lions' weak assumptions we do not have the existence of the flow generated by the velocity vector field.

3.1.3 The $B^s_{p,r}$-framework

Theorem 3.2 (Solvability of the transport equation in $B^s_{p,r}$)**.** *Let* s, p, r *satisfy* (2.21)*:*

$$s > \max\{-\frac{d}{p}, -\frac{d}{p'}\} \text{ with } \frac{1}{p'} = 1 - \frac{1}{p},$$
$$\text{or } s > -1 + \max\{-\frac{d}{p}, -\frac{d}{p'}\} \text{ if } \mathrm{div}\, v := \sum_{j=1}^d \partial_{x_j} v_j = 0.$$

Assume the initial data $\rho_0 \in B^s_{p,r}$, *the source term* $f \in L^1([0,T]; B^s_{p,r})$, *and the vector field* $v \in L^1([0,T]; \mathrm{Lip}\,) \cap L^q([0,T]; B^M_{\infty,\infty})$ *for some* $q > 1$ *and* $M > 1 - s$ *such that* $V(T) < \infty$ *with*

$$V(t) = \begin{cases} \int_0^t \|\nabla v\|_{B^{\frac{d}{p}}_{p,\infty} \cap L^\infty}, & \text{if } s < 1 + \frac{d}{p}, \\ \int_0^t \|\nabla v\|_{B^{s-1}_{p,r}}, & \text{if } s > 1 + \frac{d}{p} \text{ or } (s,r) = (1 + \frac{d}{p}, 1). \end{cases}$$

Then the equation (T) *has a unique solution* $\rho \in L^\infty([0,T]; B^s_{p,r})$ *in the distribution sense, that is,*

$$\begin{aligned} &- \int_0^T (\rho, \partial_t \varphi)_{\mathcal{D}'(\mathbb{R}^d), \mathcal{D}(\mathbb{R}^d)} \,\mathrm{d}t \, - (\rho_0, \varphi(0,x))_{\mathcal{D}'(\mathbb{R}^d), \mathcal{D}(\mathbb{R}^d)} \\ &+ \int_0^T (\mathrm{div}\,(v\rho) - (\mathrm{div}\, v)\rho, \varphi)_{\mathcal{D}'(\mathbb{R}^d), \mathcal{D}(\mathbb{R}^d)} \,\mathrm{d}t \\ &= \int_0^T (f, \varphi)_{\mathcal{D}'(\mathbb{R}^d), \mathcal{D}(\mathbb{R}^d)} \,\mathrm{d}t\,, \quad \forall \varphi \in C^\infty_c([0,T) \times \mathbb{R}^d). \end{aligned} \tag{3.45}$$

Furthermore, for a.e. $t \in [0,T]$,

$$\|\rho(t)\|_{B^s_{p,r}} \leqslant e^{CV(t)} \Big(\|\rho_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(t')} \|f(t')\|_{B^s_{p,r}} \,\mathrm{d}t' \Big), \tag{3.46}$$

and if $r < \infty$ *then* $\rho \in C([0,T]; B^s_{p,r})$ *and if* $r = \infty$ *then* $\rho \in C([0,T]; B^{s'}_{p,r})$ *for any* $s' < s$.

Proof. **Step 1 A priori estimate for general solutions** $\rho \in L^\infty([0,T]; B^s_{p,r})$
Let $\rho \in L^\infty([0,T]; B^s_{p,r})$ be a solution of (T). Let us apply the operator Δ_j to the transport equation (T) to arrive at the equation (3.27). Then if $p < \infty$, by multiplying both sides of the equation by $\mathrm{sgn}(\Delta_j\rho)|\Delta_j\rho|^{p-1}$ and integrating with respect to $x \in \mathbb{R}^d$ yields

$$\|\Delta_j\rho(t)\|_{L^p} \leqslant \|\Delta_j\rho_0\|_{L^p} + \int_0^t \big(\|\Delta_j f\|_{L^p} + \|[v\cdot\nabla, \Delta_j]\rho\|_{L^p} + \frac{1}{p}\|\mathrm{div}\, v\|_{L^\infty}\|\Delta_j\rho\|_{L^p}\big)\,\mathrm{d}t'. \tag{3.47}$$

Obviously the above holds true for $p = \infty$ (by e.g. Theorem 3.1).
We hence derive from the commutator estimate in Theorem 2.3 that

$$\|\rho(t)\|_{B^s_{p,r}} \leqslant \|\rho_0\|_{B^s_{p,r}} + \int_0^t \big(\|f\|_{B^s_{p,r}} + C(\frac{d}{dt}V)\|\rho\|_{B^s_{p,r}}\big)\,\mathrm{d}t',$$

and the estimate (3.46) follows from the Gronwall's lemma. This estimate already implies the uniqueness result.

[26.01.2022]

[28.01.2022]

Step 2 Approximate solution sequence
Let us regularize the data for the transport equation as follows:

$$\rho_{0,n} = S_n\rho_0 \in B^\infty_{p,r}, \quad f_n = \Phi_n *_t (S_n f) \in C([0,T]; B^\infty_{p,r}),$$
$$v_n = \Phi_n *_t (S_n v) \in C_b([0,T]\times\mathbb{R}^d) \text{ such that } \nabla v_n \in C([0,T]; B^\infty_{p,r})$$

where $\Phi_n = \Phi(t)$ is a mollifier sequence with respect to the time variable. Then the regularized equation

$$\partial_t\rho_n + v_n\cdot\nabla\rho_n = f_n, \quad (\rho_n)|_{t=0} = \rho_{0,n},$$

has a unique solution $\rho_n \in C([0,T]; B^\infty_{p,r})$:

$$\rho_n(t,x) = \rho_{0,n}(\psi^{-1}_{n,t}(x)) + \int_0^t f_n(t', \psi_{n,t'}(\psi^{-1}_{n,t}(x)))\,\mathrm{d}t',$$

in the distribution sense (3.45), where $\psi_{n,t} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is the flow of the vector field v_n with $\psi_{n,t} - \mathrm{Id} \in C([0,T]; B^\infty_{p,r}(\mathbb{R}^d;\mathbb{R}^d))$. Then by the estimate (3.46), we have the following estimate for ρ_n:

$$\|\rho_n(t)\|_{B^s_{p,r}} \leqslant e^{CV_n(t)}\Big(\|\rho_{n,0}\|_{B^s_{p,r}} + \int_0^t e^{-CV_n(t')}\|f_n(t')\|_{B^s_{p,r}}\,\mathrm{d}t'\Big),$$

and hence the uniform estimate for $\{\rho_n\}$:

$$\|\rho_n\|_{L^\infty([0,T];B^s_{p,r})} \leqslant e^{CV(T)}\Big(\|\rho_0\|_{B^s_{p,r}} + \int_0^T \|f(t')\|_{B^s_{p,r}}\,\mathrm{d}t'\Big).$$

Step 3: Convergence of the approximate solution sequence
In order to show the convergence of the approximate solution sequence, we prove its compactness with respect to the time variable. To this end, we will show $\partial_t\rho_n$ is uniformly bounded in $L^q([0,T];B^m_{p,\infty})$ for some m small enough, $q>1$ and $T<\infty$. However, as there is a force term $f\in L^1([0,T];B^s_{p,r})$, we would like to first consider instead

$$\widetilde{\rho_n} := \rho_n - \int_0^t f_n,$$

such that

$$\partial_t\widetilde{\rho_n} = \partial_t\rho_n - f_n = -v_n\cdot\nabla\rho_n.$$

By the above uniform estimate on $\|\rho_n\|_{L^\infty([0,T];B^s_{p,r})}$ and the assumption $v\in L^q([0,T];B^M_{\infty,\infty})$, $M>1-s$, we derive from the estimates for paraproduct and remainder the uniform bound on $\|v_n\cdot\nabla\rho_n\|_{L^q([0,T];B^m_{p,\infty})}$, $m=\min\{s-1+M, s-1, M\}$:

$$\|v_n\cdot\nabla\rho_n\|_{L^q([0,T];B^m_{p,\infty})} \leqslant C\|v_n\|_{L^q([0,T];B^M_{\infty,\infty})}\|\nabla\rho_n\|_{L^\infty([0,T];B^{s-1}_{p,r})}.$$

Therefore

$$\widetilde{\rho_n} = \rho_n - \int_0^t f_n = \rho_{0,n} + \int_0^t v_n\cdot\nabla\rho_n$$

is uniformly bounded in $L^\infty([0,T];B^s_{p,r})$ and $W^{1,q}([0,T];B^m_{p,\infty})$. By the compact embedding $W^{1,q}([0,T])\hookrightarrow\hookrightarrow C([0,T])$ for some $q>1$ and $T<\infty$ and the compactness of the multiplication operator [4]

$$M_\varphi : B^s_{p,r}\mapsto B^m_{p,\infty} \text{ via } g\mapsto\varphi g, \quad \forall\varphi\in C_0^\infty(\mathbb{R}^d),$$

the sequence $\{\varphi\widetilde{\rho_n}\}$ is compact in $C([0,T];B^m_{p,\infty})$ and hence there exists a subsequence (still denoted by (ρ_n, f_n)) such that $\widetilde{\rho_n}\to\widetilde{\rho}$ in $C([0,T];\mathcal{S}')$ and $\varphi\widetilde{\rho_n}\to\varphi\widetilde{\rho}$ in $C([0,T];B^m_{p,\infty})$ and hence in $C([0,T];B^{s'}_{p,r})$, $\forall s'<s$. Thus $\forall s'<s$,

$$\varphi\rho_n = \varphi\widetilde{\rho_n} + \int_0^t \varphi f_n \to \varphi\widetilde{\rho} + \int_0^t \varphi f =: \varphi\rho \text{ in } C([0,T];B^{s'}_{p,r}), \quad \forall\varphi\in C_0^\infty(\mathbb{R}^d).$$

[4] We accept this fact that the multiplication operator $M_\varphi : B^s_{p,\infty}\mapsto B^{s'}_{p,1}$, $s'<s$ is compact for any $\varphi\in\mathcal{S}(\mathbb{R}^d)$. See Theorem 2.94 in the book [?] for more details.

Hence the limit ρ satisfies (T) at least in the distribution sense (3.45). Finally by Fatou's property $\rho \in L^\infty([0,T]; B^s_{p,r})$ and it satisfies the estimate (3.46). Furthermore, as $\partial_t \rho$ is bounded in $L^1([0,T]; B^m_{p,\infty})$, $\rho \in C([0,T]; B^m_{p,\infty})$ and hence in $C([0,T]; B^{s'}_{p,r})$ for all $s' < s$ by interpolation.

Step 4: More properties of the solution

In order to show $\rho \in C([0,T]; B^s_{p,r})$ if $r < \infty$, we simply use $S_j\rho \in C([0,T]; B^\infty_{p,r})$ to approximate ρ in $L^\infty([0,T]; B^s_{p,r})$: We estimate the difference $\rho - S_j\rho = \sum_{k\geqslant j} \Delta_k \rho$ by use of the (3.47) as

$$\|\Delta_k\rho\|_{L^p} \leqslant e^{V(t)} \Big(\|\Delta_k\rho_0\|_{L^p} + \int_0^t \|\Delta_k f\|_{L^p} + C\|\rho\|_{L^\infty([0,T];B^s_{p,r})} \int_0^t c_k(t')\,\mathrm{d}t' \Big),$$

for some $c_k(t) \in L^1([0,T]; \ell^r)$. Therefore if $r < \infty$ then

$$\Big\| \Big(2^{ks} \big(\|\Delta_k\rho_0\|_{L^p} + \int_0^t \|\Delta_k f\|_{L^p} \big) \Big)_{k\geqslant j} \Big\|_{\ell^r}, \quad \Big\| \Big(2^{ks} \int_0^t c_k(t') \Big)_{k\geqslant j} \Big\|_{\ell^r} \to 0,$$

as $j \to \infty$. This implies $\|\rho - S_j\rho\|_{L^\infty([0,T];B^s_{p,r})} \to 0$ as $j \to \infty$ and hence $\rho \in C([0,T]; B^s_{p,r})$. □

Remark 3.3.

- *In particular if $p = r = 2$, then the H^s-regularity of ρ can be transported by the velocity vector field v with $\nabla v \in L^1([0,T]; H^\sigma \cap L^\infty)$, provided with*

$$-\tfrac{d}{2} < s < 1 + \tfrac{d}{2} \text{ and } \sigma = \tfrac{d}{2}, \quad \text{or } s = \sigma + 1 > 1 + \tfrac{d}{2}.$$

- *Compared with Theorem 3.1, for $s = 0$, here we ask more assumptions on the velocity vector field*

$$v \in L^q([0,T]; B^M_{\infty,\infty}), \quad \nabla v \in L^1([0,T]; B^{\frac{d}{p}}_{p,\infty} \cap L^\infty)$$

with $M > 1$ if $s = 0$. Nevertheless, we derive the a priori estimates (3.46) *straightforward by commutator estimates, while we used the regularisation arguments to derive* (3.43). *(Indeed the commutator estimates arguments are regularisation arguments).*

- *It is straightforward to use Fatou's property in Theorem 2.1 (with respect to the nonhomogeneous version) to derive the (weak w.r.t the time variable) compactness of the approximated solution sequence (ρ_n) in $L^\infty([0,T]; \mathcal{S}'(\mathbb{R}^d))$, such that the limit $\rho \in L^\infty([0,T]; B^s_{p,r}(\mathbb{R}^d))$ exists.*

Nevertheless we need the compactness w.r.t space variable of (ρ_n) *in* $B^{s'}_{p,r}(\mathbb{R}^d)$ *with* $M > 1 - s'$ *to ensure the convergence (in some weak sense) of*

$$\operatorname{div}(v_n\rho_n) - (\operatorname{div} v_n)\rho_n,$$

and this can be easily done by considering the multiplication operator M_φ, $\varphi \in \mathcal{D}(\mathbb{R}^d)$, *which is compact from* $B^s_{p,r}(\mathbb{R}^d)$ *to* $B^{s'}_{p,r}(\mathbb{R}^d)$ *for any* $s' < s$.

On the other side, we can relax the condition on the velocity $v \in L^1([0,T]; \mathrm{Lip} \cap B^M_{\infty,\infty})$, $M > 1 - s$ *with* $V(T) < \infty$ *in the theorem, since the weak compactness w.r.t. the time variable is sufficient to show the above convergence.*

3.2 Navier-Stokes equations

In this subsection we consider the initial value problem for the Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{NS}$$

where the time variable $t \geqslant 0$, the space variable $x \in \mathbb{R}^d$, the unknown velocity vector field $u = u(t,x) : [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and the unknown pressure term $p = p(t,x) : [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}$.
The Navier-Stokes equation can model the evolution of the (homogeneous) incompressible fluids (including liquid and gas) at comparably slow movement speed (e.g. with Mach number less than 0.3), where

- The material derivative $\partial_t + u \cdot \nabla = \partial_t + \sum_{k=1}^d u^k \partial_{x_k}$ corresponds to the transport of the fluid along the velocity vector field u, that is, $\big((\partial_t + u \cdot \nabla)\phi\big)(t, \psi_t(x)) = \frac{d}{dt}\big(\phi(t, \psi_t(x))\big)$ where $\psi = \psi(t,x) : [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is the flow of the velocity vector field u;

- The second order derivative term $-\Delta u = -\sum_{k=1}^d \partial_{x_k x_k} u$ describes the viscosity effect in the fluid;

- The divergence free condition $\operatorname{div} u = \sum_{k=1}^d \partial_{x_k} u^k = 0$ corresponds to the incompressibility of the fluid by view of (3.28):

$$\frac{d}{dt}(\det(\nabla\psi)) = 0, \text{ that is, } \det(\nabla\psi) = 1.$$

- The unknown pressure term $\nabla p = (\partial_{x_j} p)_j$ (not necessarily the physical pressure in the fluid) can be simply viewed as a Lagrangian multiplier associated to the divergence free constraint.

[28.01.2022]

[02.02.2022]

If the solution u is not smooth enough, the term $u \cdot \nabla u$ in (NS) will always be understood as $\operatorname{div}(u \otimes u)$ with the matrix $(u \otimes u)_{jk} := (u^j u^k)_{jk}$ in the distribution sense (similarly as the term $v \cdot \nabla \rho$ in the transport equation in Remark 3.1). Indeed, if u is smooth, then thanks to $\operatorname{div} u = 0$,

$$(u \cdot \nabla u) = \sum_{k=1}^{d} u^k (\partial_{x_k} u^j),$$

$$\operatorname{div}(u \otimes u) = \sum_{k=1}^{d} \partial_{x_k}(u^j u^k) = \sum_{k=1}^{d} (u^j \partial_k u^k + \partial_k u^j u^k) = u^j \operatorname{div} u + u \cdot \nabla u^j = (u \cdot \nabla u).$$

Notice that we need more regularity assumption on u in oder to make sense of the term $u \cdot \nabla u$, than of the term $u \otimes u$.

3.2.1 Weak solutions

It is straightforward to deduce the energy equality for (NS) if the solution is regular enough (say $u \in C^1([0,\infty); (H^\infty(\mathbb{R}^d))^d)$, $\nabla p \in C([0,\infty), (H^\infty(\mathbb{R}^d))^d)$). Let us take $L^2(\mathbb{R}^d)$ inner product between the equation (NS) and u itself, and we calculate the resulting terms one by one:

- $\int_{\mathbb{R}^d} \partial_t u \cdot u = \int_{\mathbb{R}^d} \frac{1}{2} \partial_t(|u|^2) = \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^d} |u|^2$,
- $\int_{\mathbb{R}^d} u \cdot \nabla u \cdot u = \int_{\mathbb{R}^d} \frac{1}{2} u \cdot \nabla(|u|^2) = -\int_{\mathbb{R}^d} \frac{1}{2}(\operatorname{div} u)|u|^2 = 0$,
- $\int_{\mathbb{R}^d} -\Delta u \cdot u = \int_{\mathbb{R}^d} |\nabla u|^2$;
- $\int_{\mathbb{R}^d} \nabla p \cdot u = -\int_{\mathbb{R}^d} p \cdot \operatorname{div} u = 0$.

Thus we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u|^2 + \int_{\mathbb{R}^d} |\nabla u|^2 = 0,$$

which implies immediately the energy equality by integration in time:

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}t' = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{3.48}$$

Thanks to the above energy estimate, J. Leray proved the global-in-time existence of the weak solution to (NS) in 1934:

Theorem 3.3 (Existence of weak solutions of (NS), J. Leray 1934)**.** *Let u_0 be a divergence-free vector field in $(L^2(\mathbb{R}^d))^d$. Then there exists a weak solution $u \in L^\infty(\mathbb{R}^+; (L^2(\mathbb{R}^d))^d) \cap L^2(\mathbb{R}^+; (\dot{H}^1(\mathbb{R}^d))^d)$ of* (NS) *satisfying the energy inequality:*

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)}\, \mathrm{d}t' \leqslant \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{3.49}$$

Here a weak solution $u \in L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^d)$ of (NS) *means that the following equality holds*

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\cdot\varphi(t,x)\,\mathrm{d}x &= \int_0^t\int_{\mathbb{R}^d} (u\cdot\partial_t\varphi + u\otimes u : \nabla\varphi + u\cdot\Delta\varphi)\,\mathrm{d}x\,dt \\ &\quad + \int_{\mathbb{R}^d} u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x\,, \end{aligned}\tag{3.50}$$

for all $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R}^d;\mathbb{R}^d)$ with $\operatorname{div}\varphi = 0$.

Ideas of the proof. The proof could be similar as the proof of Theorem 3.1 (and Theorem 3.2), and we just repeat the ideas briefly.

- Step 1 A priori estimates (already done in (3.48))
- Step 2 Construction of smooth solutions $(u_n) \subset C([0,\infty); (L^2(\mathbb{R}^d))^d)$ of the regularised equations:

  $$\partial_t u_n + P_n P(u_n\cdot\nabla u_n) - \Delta u_n = 0, \quad u_n(0,x) = P_n u_0(x), \tag{3.51}$$

  where $P = \mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}$ denotes the Leray-Helmholtz projector (see also (3.53) below), and the operator $P_n = 1_{B_n}(D)$, $n \in \mathbb{N}$ is the (rough) low-frequency cut-off operator.

- Step 3 Convergence by uniform bounds and compactness. As u_n satisfies (3.48) and the Fourier transform (with respect to x-variable) of the approximated solutions (u_n) lies in the ball B_n, we know $u_n \in C([0,\infty); (H^\infty(\mathbb{R}^d))^d)$ and uniformly bounded in $L^\infty([0,\infty); (L^2(\mathbb{R}^d))^d) \cap L^2([0,\infty); (\dot{H}^1(\mathbb{R}^d))^d)$, and hence (by Sobolev embeddings and Hölder's inequality) uniformly bounded in $L^4([0,\infty); (L^4(\mathbb{R}^2))^2)$ or $L^{\frac{8}{3}}([0,\infty); (L^4(\mathbb{R}^3))^3)$. This implies the uniform bound of $(\partial_t u_n)$ in $L^2([0,\infty); (H^{-1}(\mathbb{R}^2))^2)$ or $L^{\frac{4}{3}}([0,T]; (H^{-1}(\mathbb{R}^3))^3)$ for any fixed positive time $T > 0$. By the compactness in both time and space variables we derive the strong convergence of $u_n\cdot\phi$ to $u\phi$ in $C([0,T]; (L^2(\mathbb{R}^d))^d)$ for any fixed test function ϕ. This ensures the convergence of the integrals in (3.50).

- Step 4 Final check in two dimensional case. By the regularisation argument we can derive the energy equality (3.48), the uniqueness and the continuity of the above weak solutions. (by O. Ladyzhenskaya 1959).

Remark 3.4. *In three dimensional case, only the energy inequality* (3.49) *can hold for the weak solutions, while the energy equality* (3.48) *holds only under more regularity assumptions.*

3.2.2 Strong solutions

We first observe the following scaling invariance property of the Navier-Stokes equations (NS): If $(u, p)(t, x)$ is a solution of (NS) with the initial data u_0 on the time interval $[0, T]$, then the rescaled pair

$$(u_\lambda, p_\lambda)(t, x) = (\lambda u, \lambda^2 p)(\lambda^2 t, \lambda x), \quad \lambda > 0, \tag{3.52}$$

is a solution of (NS) of the initial data $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$ on the time interval $[0, \lambda^{-2}T]$. We calculate the $L^p(\mathbb{R}^d)$-norm of $u_{0,\lambda}$:

$$\|u_{0,\lambda}\|_{L^p(\mathbb{R}^d)} = \lambda^{1-\frac{d}{p}} \|u_0\|_{L^p(\mathbb{R}^d)}.$$

Heuristically, we then divide the exponent p of the Lebesgue space L^p into three cases:

- $p > d$ (subcritical case)
  As $\lambda \to 0$, $\|u_{0,\lambda}\|_{L^p(\mathbb{R}^d)} \to 0$ and the rescaled solution u_λ exists on the time interval $[0, \lambda^{-2}T]$ with $\lambda^{-2}T \to \infty$. This is the most favourable situation in well-posedness issue: we can make both the small initial norm and the long time interval at the same time.

- $p = d$ (critical case)
  It is easy to see that the $L^p(\mathbb{R}^d)$-norm is invariant under the scaling: $\|u_{0,\lambda}\|_{L^d(\mathbb{R}^d)} = \|u_0\|_{L^d(\mathbb{R}^d)}$, and as $\lambda \to 0$ the rescaled existing time interval is still $[0, \lambda^{-2}T]$ with $\lambda^{-2}T \to \infty$. This is always a unclear situation.

- $p < d$ (supercritical case)
  In this case as $\lambda \to 0$, $\|u_{0,\lambda}\|_{L^p(\mathbb{R}^d)} \to \infty$ as $\lambda^{-2}T \to \infty$, that is, the growing norm corresponds to longer time interval. Blowup may happen in this situation.

In the two dimensional case $d = 2$, we have established the global-in-time well-posedness results for the weak solutions in the critical case with $u_0 \in$

$(L^2(\mathbb{R}^2))^2$. While in the three dimensional case $d = 3$, the weak solutions are indeed in the supercritical case, and we do not expect the uniqueness results for the weak solutions with $u_0 \in (L^2(\mathbb{R}^3))^3$. We are going to consider the critical case $u_0 \in (L^3(\mathbb{R}^3))^3$ in dimension three.

[02.02.2022]

[09.02.2022]

It is also convenient to rewrite the equation (NS) by eliminating the pressure term ∇p. Indeed, as $\operatorname{div} u = 0$, we introduce the projection operator P:

$$P = (\mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}\,), \text{ i.e. } (\widehat{Pv})^j(\xi) = \widehat{v^j} - \sum_{k=1}^{d} \frac{\xi_j \xi_k}{|\xi|^2}\widehat{v^k} = \sum_{k=1}^{d} (\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})\widehat{v^k}, \tag{3.53}$$

such that

$$Pu = u, \quad P(\nabla p) = 0.$$

Hence we apply the operator P to the equation (NS) to arrive at

$$\begin{cases} \partial_t u - \Delta u = Q(u,u), \\ u|_{t=0} = u_0, \end{cases} \tag{PNS}$$

where

$$\widehat{Q(u,u)}^{\,j} = -P\widehat{\mathrm{div}\,(u\otimes u)}^{\,j} = -\sum_{k,l=1}^{d} (\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})(i\xi_l)\widehat{u^k u^l}.$$

More generally, the bilinear operator Q reads as

$$Q(v,w) = -\frac{1}{2}P(\mathrm{div}\,(v\otimes w) + \mathrm{div}\,(w\otimes v)),$$
$$\text{i.e. } (\widehat{Q(v,w)})^j = -\frac{1}{2}\sum_{k=1}^{d}\sum_{l=1}^{d}(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})(i\xi_l)(\widehat{v^k w^l + v^l w^k}).$$

We can show the local-in-time well-posedness result of (PNS) in different functional frameworks and here we will follow Kato's L^p approach to show the well-posedness result of (PNS) in $L^3(\mathbb{R}^3)$ in three dimensional case.

Theorem 3.4. *Let $u_0 \in (L^3(\mathbb{R}^3))^3$. Then there exists a positive time T and a unique solution $u \in C([0,T]; (L^3(\mathbb{R}^3))^3)$ of the initial value problem* (PNS). *There exists a positive constant c such that if $\|u_0\|_{L^3} \leqslant c$ then T can be chosen as $+\infty$.*

We follow a standard procedure to show the existence and uniqueness of strong solutions:

- Step 1 A priori estimates
- Step 2 Existence (and sometimes also uniqueness) by Banach fixed point theorem (It can be compared with Step 2-Step 3 in the proof of the existence of 'weak solutions' in Theorem 3.3)
- Step 3 Final check of further properties. (Be careful that the uniqueness result may be different from the uniqueness result in Step 2 above.)

Proof. **Step 1 A priori estimate**
Let us take the Fourier transform of the semilinear heat equation (PNS) and then apply Duhamel's formula to arrive at

$$\hat{u}^j(t,\xi) = e^{-t|\xi|^2}\hat{u_0}^j(\xi) - \sum_{k,l=1}^{d} \int_0^t e^{-(t-t')|\xi|^2}(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})(i\xi_l)\widehat{u^k(t')u^l}(t')\,\mathrm{d}t'.$$

Denote

$$\Gamma^j_{kl}(t,\cdot) = (2\pi)^{-\frac{3}{2}}\mathcal{F}^{-1}\Big(-e^{-t|\xi|^2}(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})(i\xi_l)\Big),$$

then the solution u reads as

$$u(t,x) = e^{t\Delta}u_0 + \sum_{k,l=1}^{d}\int_0^t \Gamma_{kl}(t-t',\cdot) * (u^k u^l)(t',\cdot)\,\mathrm{d}t'. \tag{3.54}$$

As

$$e^{t\Delta}u_0 = \mathcal{F}^{-1}\big(e^{-t|\xi|^2}\widehat{u_0}(\xi)\big) = (4\pi t)^{-\frac{3}{2}}e^{-\frac{|\cdot|^2}{4t}} * u_0,$$

we apply Young's inequality to derive for $\beta \geqslant 3$

$$\begin{aligned}\|e^{t\Delta}u_0\|_{L^\beta(\mathbb{R}^3)} &\leqslant (4\pi t)^{-\frac{3}{2}}\|e^{-\frac{|x|^2}{4t}}\|_{L^\alpha(\mathbb{R}^3)}\|u_0\|_{L^3(\mathbb{R}^3)}, \text{ with } 1 + \frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{3},\\ &\leqslant Ct^{-\frac{3}{2}+\frac{3}{2\alpha}}\|u_0\|_{L^3(\mathbb{R}^3)} = Ct^{-\frac{1}{2}(1-\frac{3}{\beta})}\|u_0\|_{L^3(\mathbb{R}^3)}.\end{aligned}$$

For any $p \in [1,\infty]$, $T \in (0,\infty)$, we define the norm

$$\|u\|_{K_p(T)} = \sup{}_{t\in(0,T]} t^{\frac{1}{2}(1-\frac{3}{p})}\|u(t)\|_{L^p(\mathbb{R}^3)}, \tag{3.55}$$

then

$$\|e^{t\Delta}u_0\|_{K_\beta(T)} \leqslant C\|u_0\|_{L^3(\mathbb{R}^3)}, \quad \forall \beta \geqslant 3. \tag{3.56}$$

Now we focus on Γ^j_{kl}. We can rewrite Γ^j_{kl} as

$$\Gamma^j_{kl}(t,\cdot) = (2\pi)^{-\frac{3}{2}}\mathcal{F}^{-1}\Big(-e^{-t|\xi|^2}\delta_{j,k}\sum_{m=1}^{d}\frac{i\xi_m^2\xi_l}{|\xi|^2}\Big) + (2\pi)^{-\frac{3}{2}}\mathcal{F}^{-1}\Big(e^{-t|\xi|^2}\frac{i\xi_j\xi_k\xi_l}{|\xi|^2}\Big).$$

By use of Fourier transform we arrive at the following pointwise bound for Γ (**Exercise.** Noticing the same structure in the first term and the second term in the definition of Γ):

$$|\Gamma^j_{kl}| \leqslant C\min\{|x|^{-4}, t^{-2}\} \leqslant C(|x|+\sqrt{t})^{-4}.$$

Hence

$$\|\Gamma^j_{kl}(t,\cdot)\|_{L^\alpha(\mathbb{R}^3)} \leqslant C\Big(\int_0^{\sqrt{t}} (t^{-2})^\alpha r^2 \,\mathrm{d}r + \int_{\sqrt{t}}^\infty (r^{-4})^\alpha r^2 \,\mathrm{d}r\Big)^{\frac{1}{\alpha}} \leqslant Ct^{-2+\frac{3}{2\alpha}}, \quad \forall \alpha \in [1,\infty].$$

Therefore Young's inequality ensures for any $(p,q) \in [1,\infty]^2$ with $\frac{1}{p}+\frac{1}{q} \in [\frac{1}{\beta}, 1]$,

$$\begin{aligned}
&\Big\|\int_0^t \Gamma_{kl}(t-t',\cdot) * (u^k u^l)(t',\cdot)\,\mathrm{d}t'\Big\|_{L^\beta(\mathbb{R}^3)} \\
&\leqslant C\int_0^t (t-t')^{-2+\frac{3}{2}(1+\frac{1}{\beta}-\frac{1}{p}-\frac{1}{q})} \|u(t')\|_{L^p(\mathbb{R}^3)} \|u(t')\|_{L^q(\mathbb{R}^3)}\,\mathrm{d}t' \\
&\leqslant C\int_0^t (t-t')^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{\beta}-\frac{1}{p}-\frac{1}{q})} (t')^{-1+\frac{3}{2}(\frac{1}{p}+\frac{1}{q})}\,\mathrm{d}t' \\
&\qquad \times \big(\sup\nolimits_{t'\geqslant 0} (t')^{\frac{1}{2}(1-\frac{3}{p})} \|u(t')\|_{L^p(\mathbb{R}^3)}\big)\big(\sup\nolimits_{t'\geqslant 0} (t')^{\frac{1}{2}(1-\frac{3}{q})} \|u(t')\|_{L^q(\mathbb{R}^3)}\big).
\end{aligned}$$

If $\frac{1}{3}+\frac{1}{\beta} > \frac{1}{p}+\frac{1}{q}$, we can control the above time integral by

$$\begin{aligned}
&Ct^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{\beta}-\frac{1}{p}-\frac{1}{q})}\int_0^{t/2} (t')^{-1+\frac{3}{2}(\frac{1}{p}+\frac{1}{q})}\,\mathrm{d}t' + Ct^{-1+\frac{3}{2}(\frac{1}{p}+\frac{1}{q})}\int_{t/2}^t (t-t')^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{\beta}-\frac{1}{p}-\frac{1}{q})}\,\mathrm{d}t' \\
&\leqslant Ct^{-\frac{1}{2}+\frac{3}{2\beta}} = Ct^{-\frac{1}{2}(1-\frac{3}{\beta})}.
\end{aligned}$$

Then we have arrived at

$$\begin{aligned}
&\Big\|\int_0^t \Gamma_{kl}(t-t',\cdot) * (u^k u^l)(t')\,\mathrm{d}t'\Big\|_{K_\beta(T)} \leqslant C\|u\|_{K_p(T)}\|u\|_{K_q(T)}, \\
&\quad \text{if } \tfrac{1}{\beta} \leqslant \tfrac{1}{p}+\tfrac{1}{q} < \tfrac{1}{3}+\tfrac{1}{\beta},\ \tfrac{1}{p}+\tfrac{1}{q} \leqslant 1.
\end{aligned} \tag{3.57}$$

To conclude, we have arrived at the following a priori estimates:

$$\begin{aligned}
&\|u\|_{K_\beta(T)} \leqslant C\big(\|u_0\|_{L^3} + \|u\|_{K_p(T)}\|u\|_{K_q(T)}\big), \\
&\forall \beta \geqslant 3 \text{ s.t. } \frac{1}{\beta} \leqslant \frac{1}{p}+\frac{1}{q} < \frac{1}{3}+\frac{1}{\beta} \leqslant \frac{2}{3}, \quad \forall T > 0.
\end{aligned} \tag{3.58}$$

[09.02.2022]

Step 2 Existence & Uniqueness of the solution in $K_6(T)$
We have established the a priori estimate (3.58) for the solution (3.54) to (PNS) in Step 1. We would like to use the contraction mapping argument to show the existence of the solution in the Banach space

$$K_6(T) = \{u \in C((0,T];(L^6(\mathbb{R}^3))^3) \mid \|u\|_{K_6(T)} < \infty\}$$

under some smallness condition on the time T or on the initial data $\|u_0\|_{L^3}$. Indeed, we first rewrite (3.54) into the following form

$$u = a + B(u,u), \quad a := e^{t\Delta}u_0, \quad B(u,u) := \int_0^t \Gamma_{kl}(t-t',\cdot) * (u^k u^l)(t',\cdot)\,\mathrm{d}t'.$$

It is easy to see that if $u_0 \in L^3(\mathbb{R}^3)$, then $e^{t\Delta}u_0 \in K_6(T)$ for any $T \in (0,\infty)$. As $\Gamma \in C((0,\infty);(L^\alpha(\mathbb{R}^3))^9)$, $\forall \alpha \in [1,\infty)$, the bilinear map

$$B : K_6(T) \times K_6(T) \mapsto K_6(T), \text{ with } \|B(u,v)\|_{K_6(T)} \leqslant C\|u\|_{K_6(T)}\|v\|_{K_6(T)}.$$

For any $u_0 \in L^3(\mathbb{R}^3)$, for any $\varepsilon > 0$, there exists $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that $\|u_0 - \varphi\|_{L^3(\mathbb{R}^3)} < \varepsilon$. On the other hand, $\|e^{t\Delta}\varphi\|_{L^\infty([0,T];L^6)} \leqslant C\|\varphi\|_{L^6}$. Thus

$$\begin{aligned}\|e^{t\Delta}u_0\|_{K_6(T)} &\leqslant \|e^{t\Delta}(u_0-\varphi)\|_{K_6(T)} + \|e^{t\Delta}\varphi\|_{K_6(T)}\\ &\leqslant C\|u_0-\varphi\|_{L^3} + CT^{\frac{1}{2}(1-\frac{3}{6})}\|\varphi\|_{L^6} \leqslant C\varepsilon + CT^{\frac{1}{4}}\|\varphi\|_{L^6}.\end{aligned}$$

We can choose T sufficiently small (depending on u_0, ε) such that

$$\|e^{t\Delta}u_0\|_{K_6(T)} \leqslant C\varepsilon. \tag{3.59}$$

Therefore for $\varepsilon > 0$, $T > 0$ sufficiently small, we derive from the contraction mapping argument that there exists a unique fixed point u of the map $u \mapsto a + B(u,u)$ in the Banach space $K_6(T)$, with

$$\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}. \tag{3.60}$$

If $\|u_0\|_{L^3(\mathbb{R}^3)} < c$, then

$$\|e^{t\Delta}u_0\|_{K_6(T)} \leqslant C\|u_0\|_{L^3} \leqslant Cc, \quad \forall T \in (0,\infty).$$

Hence in the small data case that $c > 0$ is sufficiently small, there exists a unique fixed point $u \in K_6(T)$ for any $T \in (0,\infty)$, with $\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}$.

Step 3 Continuity and Uniqueness

Although we have showed in Step 2 the existence and the uniqueness of the solution $u \in K_6(T)$ such that $\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}$ is small enough, we have to prove further $u \in C([0,T];L^3)$ and the uniqueness of the solution therein.

Now $u \in K_6(T)$ with $\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}$ is the known function and we would like to show

$$u = a + \tilde{u} \in C([0,T];L^3), \text{ with } a := e^{t\Delta}u_0 \text{ and } \tilde{u} := B(u,u).$$

Obviously $a = e^{t\Delta}u_0 \in C([0,T];L^3)$. As $u \in K_6(T)$, we infer from the derivation of the estimate (3.57) (with $\beta = 3$) that $\tilde{u} = B(u,u) \in C((0,T];L^3)$ and for any $t \in (0,T)$,

$$\|\tilde{u}\|_{L^\infty([0,t];L^3)} \leqslant C\|u\|^2_{K_6(t)} \leqslant 4C\|e^{t\Delta}u_0\|^2_{K_6(t)}, \tag{3.61}$$

where the righthand side tends to zero as $t \to 0^+$ (recalling the decomposition $e^{t\Delta}u_0 = e^{t\Delta}(u_0 - \varphi) + e^{t\Delta}\varphi$). This implies the continuity of $\tilde{u}$ at time zero and hence $\tilde{u} \in C([0,T];L^3)$.

Let us turn to the proof of uniqueness. Let $u, v \in C([0,T];L^3)$ be two solutions to (PNS) and we would like to show $u = v$. To this end, we will use energy estimates in the L^2 functional framework for their difference

$$w = u - v = (e^{t\Delta}u_0 + B(u,u)) - (e^{t\Delta}u_0 + B(v,v)) = \tilde{u} - \tilde{v} \in C([0,T];L^3).$$

We first observe that by (3.57) with $p = q = 3$, $\beta = 2$, it holds

$$\|w\|_{K_2(T)} \leqslant \|\tilde{u}\|_{K_2(T)} + \|\tilde{v}\|_{K_2(T)} \leqslant C\|u\|^2_{K_3(T)} + C\|v\|^2_{K_3(T)},$$

with $\|w\|_{K_2(T)} = \sup_{t\in(0,T]} t^{-\frac{1}{4}}\|w(t)\|_{L^2}$, and hence $w \in C([0,T];L^2)$.

By the equation (PNS), w satisfies the equation

$$\begin{cases} \partial_t w - \Delta w = Q(u,u) - Q(v,v) \\ =: f = Q(e^{t\Delta}u_0, w) + Q(w, e^{t\Delta}u_0) + Q(\tilde{u}, w) + Q(w, \tilde{v}), \\ w|_{t=0} = 0. \end{cases} \tag{w}$$

By Sobolev embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) = \dot{B}^{\frac{1}{2}}_{2,2}(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{3,2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ (by Proposition 2.2 and Proposition 2.3) and $L^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$ (by duality), we have the following estimate for the bilinear operator $Q(a,b) = -\frac{1}{2}P(\operatorname{div}(a \otimes b) + \operatorname{div}(b \otimes a))$ (noticing the zero-order projection operator $P : \dot{H}^s \mapsto \dot{H}^s$):

$$\|Q(a,b)\|_{\dot{H}^{-\frac{3}{2}}} \leqslant C\|a \otimes b\|_{\dot{H}^{-\frac{1}{2}}} \leqslant C\|a \otimes b\|_{L^{\frac{3}{2}}}$$

$$\leqslant C\min\{\|a\|_{L^3}\|b\|_{L^3}, \|a\|_{L^6}\|b\|_{L^2}, \|a\|_{L^2}\|b\|_{L^6}\}.$$

Thus $f \in C([0,T]; \dot{H}^{-\frac{3}{2}})$. We take $\dot{H}^{-\frac{1}{2}}$ inner product between the w-equation and w itself to arrive at

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{\dot{H}^{-\frac{1}{2}}} + \|\nabla w\|^2_{\dot{H}^{-\frac{1}{2}}} = \langle f, w\rangle_{\dot{H}^{-\frac{3}{2}}, \dot{H}^{\frac{1}{2}}},$$

and hence

$$\frac{1}{2}\|w(t)\|^2_{\dot{H}^{-\frac{1}{2}}} + \|\nabla w\|^2_{L^2([0,t];\dot{H}^{-\frac{1}{2}})} = \int_0^t \langle f, w\rangle_{\dot{H}^{-\frac{3}{2}}, \dot{H}^{\frac{1}{2}}} \leqslant \frac{1}{2}\|w\|^2_{L^2([0,t];\dot{H}^{\frac{1}{2}})} + \frac{1}{2}\|f\|^2_{L^2([0,t];\dot{H}^{-\frac{3}{2}})},$$

which implies

$$\|w\|_{L^\infty([0,t];\dot{H}^{-\frac{1}{2}}) \cap L^2([0,t];\dot{H}^{\frac{1}{2}})} \leqslant \|f\|_{L^2([0,t];\dot{H}^{-\frac{3}{2}})}, \quad \forall t \in [0,T].$$

Finally we would like to use Gronwall's inequality to deduce $w = 0$ in $L^\infty([0,t]; \dot{H}^{-\frac{1}{2}})$, at least in small time interval $[0,t]$. Then a standard continuation argument ensures the uniqueness result on the entire time interval $[0,T]$.
To this end, we decompose f into two parts

$$\begin{aligned} &f = f_1 + f_2, \text{ with} \\ &f_1 = Q(e^{t\Delta}u_{0,1}, w) + Q(w, e^{t\Delta}u_{0,1}) + Q(\tilde{u}, w) + Q(w, \tilde{v}), \\ &f_2 = Q(e^{t\Delta}u_{0,2}, w) + Q(w, e^{t\Delta}u_{0,2}), \end{aligned}$$

with $u_0 = u_{0,1} + u_{0,2}$, and we expect that

$$\|f_1\|_{L^2([0,t];\dot{H}^{-\frac{3}{2}})} < \frac{1}{2}\|w\|_{L^2([0,t];\dot{H}^{\frac{1}{2}})}, \text{ for small time } t > 0,$$

$$\|f_2\|^2_{L^2([0,t];\dot{H}^{-\frac{3}{2}})} \text{ involves low regularity of } w \text{ in the form } C(u_0)\int_0^t \|w\|^2_{\dot{H}^{-\frac{1}{2}}}\,\mathrm{d}t'.$$

Indeed, we decompose u_0 into $u_{0,1}$ with small L^3 value: $\|u_{0,1}\|_{L^3} \leqslant c$ and $u_{0,2}$ with regular L^6 value: $u_{0,2} \in L^6$ (e.g. we can simply take $u_{0,2} = S_j u_0$ with sufficiently large j). Then if c and the time t is chosen small enough (recalling (3.61) for the smallness of $\|(\tilde{u}, \tilde{v})\|_{L^\infty([0,t];L^3)}$ for small time)

$$\begin{aligned} &\|f_1\|_{L^2([0,t];\dot{H}^{-\frac{3}{2}})} \leqslant C\Big\|\big(\|e^{t\Delta}u_{0,1}\|_{L^3} + \|\tilde{u}\|_{K_3(t)} + \|\tilde{v}\|_{K_3(t)}\big)\|w\|_{L^3}\Big\|_{L^2([0,t])} \\ &\leqslant C\big(\|e^{t\Delta}u_{0,1}\|_{L^\infty([0,t];L^3)} + \|\tilde{u}\|_{L^\infty([0,t];L^3)} + \|\tilde{v}\|_{L^\infty([0,t];L^3)}\big)\|w\|_{L^2([0,t];\dot{H}^{\frac{1}{2}})} \\ &< \frac{1}{2}\|w\|_{L^2([0,t];\dot{H}^{\frac{1}{2}})}, \end{aligned}$$

$$\|f_2\|^2_{L^2([0,t];\dot{H}^{-\frac{3}{2}})} \leqslant \int_0^t \|e^{t\Delta}u_{0,2}\|^2_{L^6}\|w\|^2_{L^2} \leqslant C\|u_{0,2}\|^2_{L^6}\int_0^t \|w\|_{\dot{H}^{-\frac{1}{2}}}\|w\|_{\dot{H}^{\frac{1}{2}}}$$
$$\leqslant \frac{1}{8}\|w\|^2_{L^2([0,t];\dot{H}^{\frac{1}{2}})} + C\|u_{0,2}\|^4_{L^6}\int_0^t \|w\|^2_{\dot{H}^{-\frac{1}{2}}}.$$

□

Remark 3.5. *We have shown the well-posedness results for the three-dimensional Navier-Stokes equations* (NS) *in the critical Lebesgue space* $(L^3(\mathbb{R}^3))^3$ *in the sense of* (PNS)*, or more precisely* (3.54)*: If* $u \in C([0,T];(L^3(\mathbb{R}^3))^3)$ *satisfies* (PNS) *(or* (3.54)*), then we apply* div *-operator to* (PNS) *to arrive at the free heat equation for* $\operatorname{div} u$ *with zero-initial data:* $(\partial_t - \Delta)(\operatorname{div} u) = 0$*,* $(\operatorname{div} u)|_{t=0} = 0$*, which implies the incompressibility condition* $\operatorname{div} u = 0$ *immediately. Furthermore, by the equation* (PNS)*,* $Pg = 0$ *for the vector field* $g := \partial_t u + \operatorname{div}(u \otimes u) - \Delta u \in \mathcal{S}'(\mathbb{R}^3;\mathbb{R}^3)$ *such that*

$$\langle g, v\rangle_{\mathcal{S}'(\mathbb{R}^3;\mathbb{R}^3),\mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)} = 0,$$

for all divergence-free vector fields v *of* $\mathcal{S}(\mathbb{R}^3;\mathbb{R}^3)$ *such that* $v = Pv$*. This implies (see e.g. Corollary 1.2.1 in Chemin's book 'Perfect Incompressible Fluids') the existence of a tempered distribution* p *such that* $g = -\nabla p$*, and hence* $(\mathrm{NS})_1$ *holds. One can show further* $\nabla p = \nabla(-\Delta)^{-1}\operatorname{div}\operatorname{div}(u \otimes u)$ *as tempered distributions.*