Exercise 1

Let $f \in L^1(\mathbb{R}^d; \mathbb{C})$. Show that its Fourier transform $\hat{f}$ is continuous and satisfies

$$\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0.$$

Exercise 2

Let $x_0, \xi_0 \in \mathbb{R}^d$, $f, g \in L^1(\mathbb{R}^d; \mathbb{C})$ and A be a real invertible $d \times d$ matrix. Show that

$$\mathcal{F}(\tau_{x_0}(f))(\xi) = e^{-ix_0\cdot\xi}\mathcal{F}(f)(\xi),$$
$$\mathcal{F}(e^{ix\cdot\xi_0}f)(\xi) = \mathcal{F}(f)(\xi - \xi_0),$$
$$\mathcal{F}(f \circ A)(\xi) = |\det A|^{-1}\mathcal{F}(f) \circ A^{-T}\xi, \text{ and in particular } \mathcal{F}(f(\lambda\cdot)) = \lambda^{-d}(\mathcal{F}(f))(\lambda^{-1}\cdot), \forall\lambda > 0,$$
$$\mathcal{F}(f * g)(\xi) = (2\pi)^{\frac{d}{2}}\mathcal{F}(f)\mathcal{F}(g),$$
$$\int_{\mathbb{R}^d} f\mathcal{F}(g)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{F}(f)g\,\mathrm{d}x\,.$$

Exercise 3

We define the metric $d(\cdot,\cdot)$ on $\mathcal{S}(\mathbb{R}^d)$:

$$d(f,g) := \sum_{k\in\mathbb{N}} 2^{-k}\frac{\|f-g\|_{k,\mathcal{S}}}{1+\|f-g\|_{k,\mathcal{S}}}.$$

Show that the space $(\mathcal{S}(\mathbb{R}^d), d(\cdot,\cdot))$ is a complete metric space and the space $\mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$ of smooth compactly supported functions is dense in it. Hence $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, $\forall p \in [1,\infty)$.