Exercise 4

1. Show that any function $h \in L^p(\mathbb{R}^d)$, for any $p \in [1, \infty]$, defines a tempered distribution $T_h \in \mathcal{S}'(\mathbb{R}^d)$ as

$$T_h(f) = \int_{\mathbb{R}^d} hf \,\mathrm{d}x\,, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

2. Let $h \in \mathcal{S}(\mathbb{R}^d)$. Let $A \in \mathbb{R}^{d\times d}$ an invertible matrix, and $f, g \in \mathcal{S}(\mathbb{R}^d)$. Show that the following equalities hold true

   (a) The translation $(\tau_{x_0} T_h)(f) = T_h(\tau_{-x_0} f)$,
   (b) The modulation $(e^{ix\cdot\xi_0} T_h)(f) = T_h(e^{ix\cdot\xi_0} f)$,
   (c) The rescaling $(T_h \circ A)(f) = |\det(A)|^{-1} T_h(f \circ A^{-1})$,
   (d) The multiplication by $x_j : (x_j T_h)(f) = T_h(x_j f)$,
   (e) The multiplication by $g : (gT_h)(f) = T_h(fg)$,
   (f) The derivative $\partial_{x_j} : (\partial_{x_j} T_h)(f) = -T_h(\partial_{x_j} f)$,
   (g) The convolution with $f : (T_h * f)(x) = T_h(f(x - \cdot))$ (is a smooth function),
   (h) The Fourier transform $\mathcal{F} : \mathcal{F}(T_h)(f) = T_h(\mathcal{F}(f))$,
   (i) The inverse Fourier transform $\mathcal{F}^{-1}(T_h)(f) = T_h(\mathcal{F}^{-1}(f)) = T_h(\mathcal{F}R(f))$.

Exercise 5

Show that the Fourier transform is an automorphism on $\mathcal{S}'(\mathbb{R}^d)$ with

$$\mathcal{F}^{-1}\mathcal{F} = \mathrm{Id} = \mathcal{F}\mathcal{F}^{-1}, \quad \mathcal{F}^{-1} = \mathcal{F} \circ R = R \circ \mathcal{F}, \quad \mathcal{F}^2 = R, \qquad \text{on } \mathcal{S}'(\mathbb{R}^d),$$

and the following rules are valid on $\mathcal{S}'(\mathbb{R}^d)$:

1. $\mathcal{F}(\tau_{x_0}(T)) = e^{-ix_0\cdot\xi}\mathcal{F}(T),\ \mathcal{F}(e^{ix\cdot\xi_0}T) = \tau_{\xi_0}\mathcal{F}(T), \quad \forall x_0, \xi_0 \in \mathbb{R}^d$,
2. $\mathcal{F}(T \circ A) = |\det A|^{-1}\mathcal{F}(T) \circ A^{-T}$, e.g. $\mathcal{F}(T(\lambda\cdot)) = \lambda^{-d}(\mathcal{F}(T))(\lambda^{-1}\cdot), \forall \lambda > 0$,
3. $\mathcal{F}(T * f) = (2\pi)^{\frac{d}{2}}\mathcal{F}(T)\mathcal{F}(f),\ \mathcal{F}(Tf) = (2\pi)^{-\frac{d}{2}}\mathcal{F}(T) * \mathcal{F}(f), \quad \forall f \in \mathcal{S}(\mathbb{R}^d)$,
4. $\mathcal{F}((\frac{1}{i}\partial_x)^\alpha T) = \xi^\alpha \mathcal{F}(T),\ \mathcal{F}(x^\alpha T) = (i\partial_\xi)^\alpha \mathcal{F}(T), \quad \forall$multiindex α.

Exercise 6

Let $x \in \mathbb{R}$. Show that $\mathcal{F}(e^{-|x|}) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\xi^2}$.