## Exercise 7

Let $x \in \mathbb{R}^d$ and $\sigma \in (0, d)$. Show that $\mathcal{F}(|x|^{-\sigma}) = c|\xi|^{\sigma-d}$ for some constant $c$.

## Exercise 8

Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a mollifier with $\int_{\mathbb{R}^d} \rho = 1$. Let $f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty)$. Show that

$$\rho_n * f \to f \text{ in } L^p(\mathbb{R}^d), \quad \rho_n(x) = n^d \rho(nx).$$

## Exercise 9

Let $u \in L^p(\mathbb{R}^d)$, $p \in [1, \infty]$ and $U(x) = \lambda^{-d} u(\lambda^{-1} x)$, $\lambda > 0$. Show that

$$\|e^{t\Delta} u\|_{L^q} = \lambda^{\frac{d}{q} - d} \|e^{\lambda^{-2} t \Delta} U\|_{L^q}.$$

## Exercise 10

Let $h$ be a real-valued measurable function. Let $f \in W^{1,1}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) \mid f' \in L^1(\mathbb{R})\}$. We define

$$I = \int_{\mathbb{R}} f(t) e^{ih(t)} \, \mathrm{d}t \, .$$

Show that

(1) If $|h'(t)| \geq \lambda > 0$ and $h'$ is monotonic,

$$|I| \leq C\lambda^{-1} \int_{\mathbb{R}} |f'| \, \mathrm{d}x \, ;$$

(2) If $h \in C^k([a, b])$, $k \geq 2$ and $|h^{(k)}| \geq \lambda > 0$,

$$|I| \leq C\lambda^{-1/k} \int_{\mathbb{R}} |f'| \, \mathrm{d}x \, .$$

In either case the constants are independent of $a$ and $b$.

## Exercise 11

We define

$$\psi \in C_c^\infty(\mathbb{R}^d) \text{ with } \mathrm{Supp}\,(\psi) \subset K \text{ a given compact set in } \mathbb{R}^d, \tag{1}$$

and a real-valued smooth function

$$\Phi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}) \text{ with } \mathrm{Supp}\,(\psi) \subset K' \text{ a given compact set including } K. \tag{2}$$

Let $\chi$ be a smooth cut-off function which is supported on the unit ball and takes the value 1 for $|x| \leq \frac{1}{2}$. Let $N, N'$ be positive integers and $C_N, C_{N'}$ be positive constants depending only on $N, N'$ and $\|\psi\|_{C^{\max\{N,N'\}}}, \|\partial^\alpha \Phi\|_{C^{\max\{N,N'\}-1}}$ with $|\alpha| = 2$.

1. Let

$$I_1(\tau) = \frac{1}{\tau^N} \int_{\mathbb{R}^d} e^{i\tau\Phi} (L^t)^N \Big( \big(1 - \chi\big(\nabla\Phi(\xi)\big)\big) \psi(\xi) \Big) \, \mathrm{d}\xi \,,$$

where the operator

$$L^t = -L + i \frac{\Delta\Phi}{|\nabla\Phi|^2} - 2i \sum_{1 \le j,k \le d} \frac{\partial_j \Phi \partial_k \Phi \partial_{jk} \Phi}{|\nabla\Phi|^4}.$$

Show that

$$|I_1(\tau)| \le \frac{C_N}{\tau^N}.$$

2. Let

$$I_2(\tau) = \int_{\mathbb{R}^d} e^{i\tau\Phi} (L_\tau^t)^{N'} \Big( \chi\big(\nabla\Phi(\xi)\big) \psi(\xi) \Big) \, \mathrm{d}\xi \,,$$

where the operator

$$L_\tau^t = \frac{1 + i \sum_{j=1}^d \partial_{\xi_j} \Phi \partial_{\xi_j}}{1 + \tau |\nabla\Phi|^2} + i \frac{\Delta\Phi}{1 + \tau|\nabla\Phi|^2} - 2i\tau \sum_{1 \le j,k \le d} \frac{\partial_j \Phi \partial_k \Phi \partial_{jk} \Phi}{(1 + \tau|\nabla\Phi|^2)^2}.$$

Show that

$$|I_2(\tau)| \le C_{N'} \int_{\{|\nabla\Phi| \le 1\}} \frac{1}{(1 + \tau|\nabla\Phi|^2)^{N'}} \, \mathrm{d}\xi \,.$$