JProf. Dr. Xian Liao
M.Sc. Zihui He

## Exercise 12

1. Let $u \in \mathcal{S}'(\mathbb{R}^d)$. Show that
$$S_j u \to u \text{ in } \mathcal{S}'(\mathbb{R}^d), \text{ as } j \to \infty,$$
where $\widehat{S_j u} = \chi(2^{-j}\cdot)\hat{u}$.
2. Let $u \in L^p(\mathbb{R}^d)$, $p \in [1, \infty)$. Show that
$$S_j u \to u \text{ in } L^p(\mathbb{R}^d), \quad p \in [1, \infty).$$

## Exercise 13

Show that the function $|x|^{-\sigma}$, $\sigma \in (0, d)$ belongs to $\dot{B}^{\frac{d}{p}-\sigma}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1, \infty]$, but not to $\dot{B}^{\frac{d}{p}-\sigma}_{p,r}(\mathbb{R}^d)$ for any $r \in [1, \infty)$.

## Exercise 14

Show the following properties for the homogeneous Besov spaces:

1. Homogeneity: $\|u(\lambda\cdot)\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} = \lambda^{s-\frac{d}{p}}\|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^d)}$, $\forall \lambda > 0$;
2. Embedding: $\dot{B}^s_{p,r}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s-d(\frac{1}{p}-\frac{1}{p_1})}_{p_1,r_1}(\mathbb{R}^d)$, if $p \leq p_1$, $r \leq r_1$ and in particular $\dot{B}^s_{p,1} \hookrightarrow \dot{B}^s_{p,r} \hookrightarrow \dot{B}^s_{p,\infty}$.

## Exercise 15

Use Fatou's property to show the convergence of a Cauchy sequence in $\dot{B}^s_{p,r}(\mathbb{R}^d)$ with $s < \frac{d}{p}$ or $(s, p, r) = (\frac{d}{p}, p, 1)$.