Winter term 2021/22 **Karlsruher Institut für Technologie**
Fourier analysis and its applications to PDEs JProf. Dr. Xian Liao
Exercise sheet 5 M.Sc. Zihui He

Exercise 16

Show that

1. $\|\cdot\|_{\dot{H}^s} \sim \|\cdot\|_{\dot{B}^s_{2,2}}$ and $\dot{H}^s \subset \dot{B}^s_{2,2}$ for any $s \in \mathbb{R}$.
2. If $s < \frac{d}{2}$, then $\dot{H}^s(\mathbb{R}^d) = \dot{B}^s_{2,2}(\mathbb{R}^d)$.

Exercise 17

Let $(p, q) \in [1, \infty]^2$. Show that

1. $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$ and more generally $\dot{B}^{\frac{d}{p}-\frac{d}{q}}_{p,1} \hookrightarrow L^q$ whenever $p \leq q$;
2. $\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$ whenever $p \in [1, \infty)$;
3. The space of bounded measures on $\mathbb{R}^d$ is continuously embedded in $\dot{B}^0_{1,\infty}$.

Exercise 18

Let $(p, r) \in [1, \infty]^2$. Show that if $\sigma < 0$, then

$$\left\| \left(2^{j\sigma} \|\dot{S}_j u\|_{L^p} \right) \right\|_{\ell^r} \sim \|u\|_{\dot{B}^\sigma_{p,r}}.$$