Winter term 2021/22 **Karlsruher Institut für Technologie**
Fourier analysis and its applications to PDEs JProf. Dr. Xian Liao
Exercise sheet 6 M.Sc. Zihui He

Exercise 19

Let $s \in [0, \frac{d}{2})$. Show that there exists a constant C such that

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2s}} \, \mathrm{d}x \le C \|f\|^2_{\dot{H}^s(\mathbb{R}^d)}.$$

Exercise 20

Let $v \in L^1_{\mathrm{loc}}(\mathbb{R}; \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^d))$. Let $\psi_t(\cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ be the unique solution of the following initial value problem

$$\partial_t \psi_t(x) = v(\psi_t(x)), \quad \psi_t(x)|_{t=0} = x.$$

Show that

$$\frac{d}{dt}(\det(\nabla \psi_t)(t,x)) = (\operatorname{div} v)(\psi_t(x)) \cdot (\det(\nabla \psi_t)(x)).$$

Exercise 21

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, $q \in [1, \infty]$. Let η be a mollifier, such that $\eta \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \eta = 1$, $\eta \ge 0$, $\mathrm{Supp}\,(\eta) \subset B_1(0)$. Let $\eta_\varepsilon = \varepsilon^{-d}\eta(\varepsilon^{-1}\cdot)$. Show that

$$\|[\eta_\varepsilon *, \phi] g\|_{L^q(\mathbb{R}^d)} \le \varepsilon \|\nabla \phi\|_{L^\infty(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}.$$