Exercise 22

Let $v \in L^1([0,T];W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$ and $\rho \in L^\infty([0,T];L^p(\mathbb{R}^d))$, $p \in [1,\infty)$. Let η be a mollifier defined as in Exercise 21. Show that

$$[v \cdot \nabla, \eta_\varepsilon *]\rho \to 0 \text{ in } L^1([0,T];L^p(\mathbb{R}^d)).$$

Exercise 23

Let $v \in L^1([0,T];W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$. Let $\psi_t(x) \in C_b([0,T];W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$ be the solution of the following ODE

$$\partial_t \psi_t(x) = v(t, \psi_t(x)), \quad \psi_t(x)|_{t=0} = x.$$

Let $\rho_0 \in L^p(\mathbb{R}^d)$, $p \in [1,\infty]$ and $f \in L^1([0,T];L^p(\mathbb{R}^d))$. We define

$$\rho(t,x) = \rho_0(\psi_t^{-1}(x)) + \int_0^t f(t', \psi_{t'}(\psi_t^{-1}(x)))dt'.$$

Show that $\rho \in L^\infty([0,T];L^p(\mathbb{R}^d))$ and satisfies the following weak formulation

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^d} \rho \partial_t \phi \, \mathrm{d}t \, \mathrm{d}x - \int_{\mathbb{R}^d} \rho_0 \phi(0,x) \, \mathrm{d}x - \int_0^T \int_{\mathbb{R}^d} \mathrm{div}\,(v\phi)\rho \, \mathrm{d}t \, \mathrm{d}x \\ &= \int_0^T \int_{\mathbb{R}^d} f\phi \, \mathrm{d}t \, \mathrm{d}x, \quad \forall \phi \in C_c^\infty([0,T) \times \mathbb{R}^d). \end{aligned}$$

Exercise 24

Let

$$\Gamma_{kl}^j(t,\cdot) = (2\pi)^{-\frac{3}{2}} \mathcal{F}^{-1}\Big(-e^{-t|\xi|^2} \delta_{j,k} \sum_{m=1}^d \frac{i\xi_m^2 \xi_l}{|\xi|^2}\Big) + (2\pi)^{-\frac{3}{2}} \mathcal{F}^{-1}\Big(e^{-t|\xi|^2} \frac{i\xi_j \xi_k \xi_l}{|\xi|^2}\Big), \quad t \geq 0.$$

Show that

$$|\Gamma_{kl}^j| \leq C(|x| + \sqrt{t})^{-4} \quad \text{and} \quad \|\Gamma_{kl}^j(t,\cdot)\|_{L^\alpha(\mathbb{R}^3)} \leq Ct^{-2+\frac{3}{2\alpha}}, \quad \forall \alpha \in [1,\infty].$$