$\boxed{6}$ (a) Beh: $\cos(z) = 0 \overset{\text{für } z\in\mathbb{C}}{\iff} z \in \left\{\frac{\pi}{2} + \pi k : k \in \mathbb{Z}\right\}$

Bew: „$\Leftarrow$" ✓ wg. kapl. Erw.

„$\Rightarrow$" Sei $z \in \mathbb{C}$ mit $\cos(z) = 0$ gegeben. Dann gilt

$$\frac{e^{iz} + e^{-iz}}{2} = 0, \text{ d.h. } e^{2iz} + 1 = 0$$

Schreibe $z = x + iy$, d.h.

$$e^{2ix - 2y} = -1$$

Hieraus ergeben sich $e^{-2y} = 1$, d.h. $y = 0$

und $\cos(2x) + i\sin(2x) = -1$, d.h. $\begin{cases} \cos(2x) = -1 \\ \sin(2x) = 0 \end{cases}$

Die Lösungen hiervon sind gerade $x = \frac{\pi}{2} + k\pi$ mit $k \in \mathbb{Z}$. ∎

(b) (i) Beh: Der Konv.radius r von $\sum_{n=0}^{\infty} (1+i^n)^{n/2} z^n$ beträgt $r = \frac{1}{\sqrt{2}}$.

Bew: $\sqrt[n]{|(1+i^n)^{n/2}|} = \sqrt{|1+i^n|} = \begin{cases} \sqrt{2}, & n \equiv 0 \mod 4 \\ \sqrt[4]{2}, & n \equiv 1,3 \mod 4 \\ 0, & n \equiv 2 \mod 4. \end{cases}$

Also $\limsup_{n\to\infty} \sqrt[n]{|1+i^n|^{n/2}} = \sqrt{2}$, d.h. $r = \frac{1}{\sqrt{2}}$ ∎

(ii) Beh: Der Konv.radius r von $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{n^{n-1}}{n!} z^n$ beträgt $\frac{1}{e}$.

Bew: Setze $a_n := \frac{(-1)^{n-1}}{n!} n^{n-1}$, dann gilt

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^n}{(n+1)!} \frac{n!}{n^{n-1}} = \frac{(n+1)^{n-1}}{\cancel{n!}} \frac{\cancel{n!}}{n^{n-1}} = \left(1 + \frac{1}{n}\right)^{n-1} \xrightarrow[n\to\infty]{} e,$$

also $r = \frac{1}{e}$. ∎

Zu $(1+i)^i$: Wegen $\log(1+i) = \log|1+i| + i\arg(1+i) = \log\sqrt{2} + i\frac{\pi}{4}$ $\quad (\arg(1+i) = \arctan(1))$

erhalten wir

$$(1+i)^i = e^{i\log(1+i)} = e^{i(\log\sqrt{2} + i\frac{\pi}{4})} = e^{i\log\sqrt{2} - \frac{\pi}{4}} = e^{-\frac{\pi}{4}}\left(\cos(\log\sqrt{2}) + i\,\sin(\log\sqrt{2})\right),$$

also $\mathrm{Re}\left[(1+i)^i\right] = e^{-\frac{\pi}{4}}\left(\cos(\log\sqrt{2})\right)$, $\mathrm{Im}\left[(1+i)^i\right] = e^{-\frac{\pi}{4}}\sin(\log\sqrt{2})$.

Zu $i^{\frac{1}{i}}$: Wegen $\frac{1}{i} = -i$ und $\log i = \log|i| + i\arg(i) = i\frac{\pi}{2}$ folgt

$$i^{\frac{1}{i}} = e^{\frac{1}{i}\log i} = e^{-i\,i\frac{\pi}{2}} = e^{\frac{\pi}{2}},$$ also $\mathrm{Re}\left(i^{\frac{1}{i}}\right) = e^{\frac{\pi}{2}}$, $\mathrm{Im}\left(i^{\frac{1}{i}}\right) = 0$.

Zu $(\log i)^i$: $\log i = i\frac{\pi}{2}$ (s.o.), also

$$\log(\log i) = \log\left(i\frac{\pi}{2}\right) = \log\left|i\frac{\pi}{2}\right| + i\arg\left(i\frac{\pi}{2}\right) = \log\frac{\pi}{2} + i\frac{\pi}{2},$$ also

$$(\log i)^i = e^{i\log(\log i)} = e^{i\log\frac{\pi}{2} - \frac{\pi}{2}} = e^{-\frac{\pi}{2}}\left(\cos\left(\log\frac{\pi}{2}\right) + i\sin\left(\log\frac{\pi}{2}\right)\right),$$

also $\mathrm{Re}\left[(\log i)^i\right] = e^{-\frac{\pi}{2}}\cos\left(\log\frac{\pi}{2}\right)$ und $\mathrm{Im}\left[(\log i)^i\right] = e^{-\frac{\pi}{2}}\sin\left(\log\frac{\pi}{2}\right)$.

Zu i^{i^i}: $\log i = i\frac{\pi}{2}$, also $i^i = e^{i\frac{i\pi}{2}} = e^{-\frac{\pi}{2}}$, damit:

$$i^{(i^i)} = i^{e^{-\frac{\pi}{2}}} = e^{e^{-\frac{\pi}{2}}\log i} = e^{i\frac{\pi}{2}e^{-\frac{\pi}{2}}} = \cos\left(\frac{\pi}{2}e^{-\frac{\pi}{2}}\right) + i\sin\left(\frac{\pi}{2}e^{-\frac{\pi}{2}}\right),$$

also $\mathrm{Re}\left(i^{i^i}\right) = \cos\left(\frac{\pi}{2}e^{-\frac{\pi}{2}}\right)$ und $\mathrm{Im}\left(i^{i^i}\right) = \sin\left(\frac{\pi}{2}e^{-\frac{\pi}{2}}\right)$.