Winter term 2022/2023 **Karlsruhe Institute of Technology**
Functional Analysis JProf. Dr. Xian Liao
Exercise sheet 1
Due date: 02.11.2022

Exercise 1

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $n \in \mathbb{N}$. Show that the following are Banach spaces.

1. $\mathbb{R}^n$ with $\mathbb{K} = \mathbb{R}$, and $\mathbb{C}^n$ with $\mathbb{K} = \mathbb{C}$, equipped with the Euclidean norm.

2. $\mathbb{B}(X) = \{f : X \to \mathbb{K} : \sup_{x\in X} |f(x)| < \infty\}$ for a set X, equipped with the supremum norm
$$\|f\|_{\mathbb{B}(X)} = \sup_{x\in X} |f(x)|.$$

3. $C_b(X) = \{f \in \mathbb{B}(X) : f \text{ continuous}\}$ for a metric space (X, d), equipped with the supremum norm.

4. $C_0(\mathbb{R}^n) = \{f \in C_b(\mathbb{R}^n) : \lim_{|x|\to\infty} |f(x)| = 0\}$ equipped with the supremum norm.

5. $c_0 = \{(x_j)_{j\in\mathbb{N}} \subset \mathbb{K} : \lim_{j\to\infty} |x_j| = 0\}$ equipped with the supremum norm
$$\|(x_j)_{j\in\mathbb{N}}\|_{l^\infty} = \sup_{j\in\mathbb{N}} |x_j|.$$

6. $C_b^k(U) = \{f \in C^k(U) : \|\partial^\alpha f\|_{\mathbb{B}(U)} < \infty \ \forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq k\}$ for an open set $U \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, with
$$\|f\|_{C_b^k(U)} = \max_{|\alpha|\leq k} \|\partial^\alpha f\|_{\mathbb{B}(U)},$$
where $|\alpha| = \alpha_1 + ... + \alpha_n$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}$ for $\alpha \in \mathbb{N}_0^n$.

7. The space of bounded holomorphic functions $H^\infty(U)$ on an open set $U \subset \mathbb{C}$, equipped with the supremum norm.

Exercise 2

1. Prove Lemma 1.8: Let X and Y be normed spaces. Their direct sum $X \oplus Y = X \times Y$ is a vector space. Then for any $1 \leq p \leq \infty$,
$$\|(x, y)\|_p = \|(|x|_X, |y|_Y)\|_{l^p}$$
defines a norm with which $X \oplus Y$ becomes a Banach space, and all the norms $\|\cdot\|_p$ are equivalent.

2. Let X be a Banach space and $U \subset X$ a closed subvector space. Show that U is a Banach space. Furthermore, prove that
$$\|\tilde{x}\|_{X/U} = \inf_{y\in U} \|y - x\|$$
defines a norm on X/U, which turns X/U into a Banach space. Here $\tilde{x}$ denotes the equivalence class of $x \in X$.