Winter term 2022/2023 | **Karlsruhe Institute of Technology**
Functional Analysis
Exercise sheet 10
Due date: 18.01.2023

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Exercise 1

Give an example of normed spaces X, Y and a subset $\mathcal{F} \subset L(X,Y)$ such that

$$\sup\{\|Tx\|_Y : T \in \mathcal{F}\} < \infty$$

for all $x \in X$, but

$$\sup\{\|T\|_{X\to Y} : T \in \mathcal{F}\} = \infty.$$

Exercise 2

Show that the set of nowhere differentiable functions in $C_b(0,1)$ is dense in $C_b(0,1)$.

Exercise 3

Let $U \subset \mathbb{R}^d$ be an open set. Show that any $f \in L^1_{\text{loc}}(U)$ defines a distribution $T_f \in \mathcal{D}'(U)$ given by

$$T_f(\varphi) = \int_U f\varphi dm^d$$

for $\varphi \in \mathcal{D}(U)$. Moreover, show that T_f is uniquely determined by f in the sense that the map $L^1_{\text{loc}}(U) \to \mathcal{D}'(U)$, $f \mapsto T_f$ is linear, continuous and injective.

Exercise 4

Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$.

1. Show that $\phi * T \in C^\infty(\mathbb{R}^d)$ and $\partial_{x_j}(\phi * T) = (\partial_{x_j}\phi) * T = \phi * (\partial_{x_j} T)$ almost everywhere on $\mathbb{R}^d$ for $j = 1, ..., N$.
2. Show that if $\psi \in L^1(\mathbb{R}^d)$ then $\phi * T_\psi(x) = \phi * \psi(x)$ for almost every $x \in \mathbb{R}^d$.
3. Show that if supp $\phi = K_1$ and supp $T = K_2$, then supp $\phi * T \subset K_1 + K_2$.

Exercise 5

1. Show that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $p \in [1,\infty)$, but not for $p = \infty$.
2. Show that $C_c(\mathbb{R}^d)$ is not dense in $C_b(\mathbb{R}^d)$, but that its closure with respect to $\|\cdot\|_{C_b}$ is given by $C_0(\mathbb{R}^d) := \{f \in C_b(\mathbb{R}^d) : \lim_{|x|\to\infty} |f(x)| = 0\}$.
3. Show that $C_c((0,1))$ is not dense in $C_b([0,1])$, but that its closure with respect to $\|\cdot\|_{C_b}$ is given by $C_0([0,1]) = \{f \in C_b([0,1]) : f(0) = f(1) = 0\}$.