Winter term 2022/2023 — **Karlsruhe Institute of Technology**

Functional Analysis — JProf. Dr. Xian Liao

Exercise sheet 11 — M.Sc. Rebekka Zimmermann

Due date: 25.01.2023

Exercise 1

Prove or disprove whether the following maps define elements $T \in \mathcal{D}'(U)$:

1. $U = (0,1)$, $T\varphi = \sum_{n=2}^{\infty} \varphi^{(n)}(\frac{1}{n})$,
2. $U = \mathbb{R}$, $T\varphi = \sum_{n=1}^{\infty} \varphi^{(n)}(\frac{1}{n})$,

where $\varphi \in \mathcal{D}(U)$.

Exercise 2

Let $U \subset \mathbb{R}^d$ be open. Suppose that $T_n \to T$ in $\mathcal{D}'(U)$ and that $K \subset U$ is compact. Show that there exist $k \in \mathbb{N}$ and $C > 0$ so that

$$\sup_{n\in\mathbb{N}} |T_n(\varphi)| \le C\|\varphi\|_{C_b^k(U)},$$

for all $\varphi \in X_K$, and

$$\sup\{|T_n(\varphi) - T(\varphi)| : \varphi \in X_K, \|\varphi\|_{C_b^k} \le 1\} \to 0$$

as $n \to \infty$.

Exercise 3

1. Show that $\delta_0 \in \mathcal{E}'(\mathbb{R}^d)$ with supp $\delta_0 = \{0\}$.
2. Show that $\delta_0 * T = T$ for all $T \in \mathcal{D}'(\mathbb{R}^d)$.
3. Let η_r be as in Exercise 2 on Exercise Sheet 9. Show that $\eta_r \to \delta_0$ as $r \to 0^+$ in the sense of distributions. Conclude that $\eta_r * T \to T$ as $r \to 0^+$ in the sense of distributions for all $T \in \mathcal{D}'(\mathbb{R}^d)$.

Exercise 4

Let η_r be as in Exercise 2 on Exercise Sheet 9.

1. Show that $\eta_r * f \to f$ in $C_b(\mathbb{R}^d)$ as $r \to 0^+$ for all $f \in C_c(\mathbb{R}^d)$, but not for all $f \in C_b(\mathbb{R}^d)$. In particular, we do not have $\eta_r * f \to f$ as $r \to 0^+$ in $L^\infty(\mathbb{R}^d)$ for all $f \in L^\infty(\mathbb{R}^d)$.
2. Let $U \subset \mathbb{R}^d$ be an open set and $k \in \mathbb{N}$. Show that for any $f \in C_c^k(U)$ there exists a compact set $K \subset U$ and $\epsilon > 0$ such that supp $(\eta_r * f) \subset K$ for all $0 < r \le \epsilon$, and $\eta_r * f \to f$ in $C_b^k(U)$ as $r \to 0^+$.

Exercise 5

Let $U \subset \mathbb{R}^d$ be an open connected set, let $T \in \mathcal{D}'(U)$ and suppose that $\partial_j T = 0$ for all $j = 1, ..., d$. Show that there exists a constant $c \in \mathbb{R}$ such that $T = T_c$.