Winter term 2022/2023
Functional Analysis
Exercise sheet 12
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Exercise 1

Let $T \in \mathcal{S}'(\mathbb{R}^d)$.

1. Show that there exist $k \in \mathbb{N}$ and $c > 0$ such that for all $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$|T\phi| \leq c\|\phi\|_{k,\mathcal{S}}.$$

2. Suppose that $(T_n)_{n\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d)$ converges to T in $\mathcal{S}'(\mathbb{R}^d)$. Show that there exist $l \in \mathbb{N}$ and $C > 0$ such that

$$\sup_{n\in\mathbb{N}} |T_n\phi| \leq C\|\phi\|_{l,\mathcal{S}},$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, and

$$\sup\{|T_n(\phi) - T\phi| \,:\, \phi \in \mathcal{S}(\mathbb{R}^d),\, \|\phi\|_{l,\mathcal{S}} \leq 1\} \to 0, \quad n \to \infty.$$

Exercise 2

Let $1 \leq p \leq \infty$. Show that the following embeddings hold

$$\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d).$$

Moreover, show that the inclusions are dense if $p < \infty$.

Exercise 3

Let $\mathcal{F}$ denote the Fourier transform as introduced in the lecture.

1. Show that the following identities hold for all $f \in \mathcal{S}(\mathbb{R}^d)$ and multiindices $\alpha \in \mathbb{N}_0^d$
   (a) $\mathcal{F}((\frac{1}{i}\partial_x)^\alpha f) = \xi^\alpha \mathcal{F}(f)$,
   (b) $\mathcal{F}(x^\alpha f) = (i\partial_\xi)^\alpha \mathcal{F}(f)$.

2. Show that $\mathcal{F}$ maps continuously from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$, i.e., that for any $k \in \mathbb{N}$ there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$\|\mathcal{F}(f)\|_{k,\mathcal{S}} \leq C\|f\|_{N,\mathcal{S}}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Exercise 4

1. Show that the Fourier transform defines a unitary operator $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, which can be given by
$$\mathcal{F}(f)(\xi) = \lim_{R\to\infty} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{B_R(0)} e^{-ix\cdot\xi} f(x)dx$$
for almost every $\xi \in \mathbb{R}^d$ and for all $f \in L^2(\mathbb{R}^d)$.

2. Show that for all $p \in [1, 2]$ the Fourier transform defines a continuous linear map $\mathcal{F} : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$, with
$$\|\mathcal{F}\|_{L^p\to L^{p'}} \leq (2\pi)^{-\frac{d}{2}(\frac{2}{p}-1)},$$
where p' is determined by $\frac{1}{p} + \frac{1}{p'} = 1$.