**Winter term 2022/2023** **Karlsruhe Institute of Technology**
Functional Analysis JProf. Dr. Xian Liao
Exercise sheet 13 M.Sc. Rebekka Zimmermann
Due date: 08.02.2023

## Exercise 1

Let $U \subset \mathbb{R}^d$ be an open bounded domain, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$.

1. Show that if $f \in C^k(\overline{U})$, then $f \in W^{k,p}(U)$ and its weak derivatives coincide with its strong derivatives almost everywhere on $U$.
2. Give an example of a function $f \in W^{1,p}(U)$ such that $f \notin C^1(U)$.
3. Give an example of a function $f \in W^{k,p}(U)$ such that $f \notin W_0^{k,p}(U)$.
4. Give an example of a domain $U \subset \mathbb{R}^d$ such that there is no linear continuous extension map $W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d)$.

## Exercise 2

Let $U \subset \mathbb{R}^d$ be open, $1 \leq p \leq \infty$, and $k \in \mathbb{N}$.

1. Show that if $g \in C_b^k(U)$ and $f \in W^{k,p}(U)$, then $gf \in W^{k,p}(U)$ with $\partial_{x_j}(fg) = g\partial_{x_j} f + f\partial_{x_j} g$ for all $j = 1, ..., d$.
2. Let $V \subset \mathbb{R}^d$ be open and let $\phi : V \to U$ a $C_b^k$ be a diffeomorphism, i.e., $\phi$ is bijective with $\phi \in C_b^k(V; U)$ and $\phi^{-1} \in C_b^k(U; V)$. Show that if $f \in W^{k,p}(U)$ then $f \circ \phi \in W^{k,p}(V)$, and there holds
$$\|f \circ \phi\|_{W^{k,p}(V)} \leq C\|f\|_{W^{k,p}(U)}$$
for some constant $C > 0$ depending only on $\|\phi\|_{C_b^k(V;U)}$ and $\|\phi^{-1}\|_{C_b^k(U;V)}$.

## Exercise 3

Let $V = \{x \in \mathbb{R} : x \leq 0\}$, $1 \leq p < \infty$ and $k \in \mathbb{N}$. For $f \in W^{k,p}(V)$, $1 \leq l \leq k$, and almost every $x \in \mathbb{R}$ we define

$$F(x) = \begin{cases} f(x), & \text{if } x < 0, \\ \sum_{j=1}^{k+1} a_j f(-jx), & \text{if } x > 0, \end{cases} \qquad F_l(x) = \begin{cases} \frac{d^l}{dx^l} f(x), & \text{if } x < 0, \\ \sum_{j=1}^{k+1} a_j (-j)^l (\frac{d^l}{dx^l} f)(-jx), & \text{if } x > 0, \end{cases}$$

where $a_1, ..., a_{k+1}$ satisfy the linear equations in the proof of Theorem 4.35. Show that

$$(-1)^l \int_{\mathbb{R}} F(x) \frac{d^l}{dx^l} \varphi(x) dx = \int_{\mathbb{R}} F_l(x) \varphi(x) dx$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$.

## Exercise 4

Show the Poincaré inequality in a ball: If $1 \leq p \leq \infty$, $R > 0$ and $f \in W^{1,p}(B_R(0))$ then

$$\|f - f_B\|_{L^p(B_R(0))} \leq C \| |\nabla f| \|_{L^p(B_R(0))},$$

for some constant $C = C(d, p, R) > 0$, where

$$f_B := \frac{1}{m^d(B_R(0))} \int_{B_R(0)} f dm^d$$

denotes the average of $f$ in $B_R(0)$.

## Exercise 5

Let $U \subset \mathbb{R}^d$ be open, $d < p < \infty$, and $\tilde{f} \in W^{1,p}(U)$. Show that there exists a continuous representative $f$ in the equivalence class of $\tilde{f}$, and there exists a constant $c_{d,p} > 0$ such that

$$|f(x) - f(y)| \leq c_{d,p} |x - y|^{1 - \frac{d}{p}} \| |\nabla f| \|_{L^p(B_{|x-y|}(x))}$$

for all $x, y \in U$ with $|x - y| < \operatorname{dist}(x, \mathbb{R}^d \backslash U)$.