Winter term 2022/2023 **Karlsruhe Institute of Technology**
Functional Analysis
Exercise sheet 14
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JProf. Dr. Xian Liao
M.Sc. Rebekka Zimmermann

Exercise 1

Recall that the space $c_0(\mathbb{N}) = \{(x_j)_{j\in\mathbb{N}} \in l^\infty(\mathbb{N}) : \lim_{j\to\infty} x_j = 0\}$ endowed with the supremum norm $\|\cdot\|_{l^\infty}$ is a Banach space. Find an isometric isomorphism $(c_0(\mathbb{N}))^* \to l^1(\mathbb{N})$.

Exercise 2

1. Let $1 < p < \infty$, $(x^{(k)})_{k\in\mathbb{N}} \subset l^p(\mathbb{N})$ and $x \in l^p(\mathbb{N})$. Show that $x^{(k)} \rightharpoonup x$ weakly in $l^p(\mathbb{N})$ if and only if $(x^{(k)})_{k\in\mathbb{N}}$ is bounded in $l^p(\mathbb{N})$ and $x_j^{(k)} \to x_j$ for all $j \in \mathbb{N}$ as $k \to \infty$.
2. Let $(x^{(k)})_{k\in\mathbb{N}} \subset l^1(\mathbb{N})$ and $x \in l^1(\mathbb{N})$. Show that $x^{(k)} \rightharpoonup x$ weakly in $l^1(\mathbb{N})$ if and only if $x^{(k)} \to x$ in $l^1(\mathbb{N})$ as $k \to \infty$.

Exercise 3

Let X be a $\mathbb{K}$ vector space and $C \subset X$ be convex.

1. Show that the Minkowski functional $p_C : X \to [0, \infty]$ defined by
$$p_C(x) = \inf\{\lambda > 0 : \frac{1}{\lambda}x \in C\}$$
for $x \in X$, is sublinear.
2. Show that if C has the property that for every $x \in X$ there exists $\lambda > 0$ so that $\lambda x \in C$, then $p_C(x) < \infty$ for all $x \in X$.
3. Suppose that X is a normed space. Show that its norm is the Minkowski functional of the unit ball, i.e., for all $x \in X$ there holds $\|x\| = p_{B_1(0)}(x)$.

Exercise 4

Let X be a normed space and $C \subset X$ an open, convex set with $0 \notin C$. Show that there exists an element $x^* \in X^*$ which satisfies $\operatorname{Re} x^*(x) < 0$ for all $x \in C$.

Exercise 5

Let $U \subset \mathbb{R}^d$ be a domain, let $k \in \mathbb{N}$ and $1 < p < \infty$. Show that the spaces $L^p(U)$ and $W^{k,p}(U)$ are uniformly convex, i.e., for all $\epsilon > 0$ there holds

$$\delta(\epsilon) := \inf\{1 - \frac{1}{2}\|f + g\|_X : f, g \in X,\ \|f\|_X, \|g\|_X \le 1,\ \|f - g\|_X > \epsilon\} > 0,$$

where $X = L^p(U)$ or $X = W^{k,p}(U)$.