Winter term 2022/2023 **Karlsruhe Institute of Technology**
Functional Analysis JProf. Dr. Xian Liao
Exercise sheet 2
Due date: 09.11.2022

Exercise 1

Let X and Y be normed spaces. Show that the set $L(X,Y)$ of continuous linear maps from X to Y is a vector space with the addition $(T_1 + T_2)(x) := T_1x + T_2x$, and multiplication $(\lambda T)(x) := \lambda Tx$, for $T, T_1, T_2 \in L(X,Y)$, $\lambda \in \mathbb{K}$ and $x \in X$.

Exercise 2

1. Let X be a normed space. Show that the addition $+ : X \times X \mapsto X$, the scalar multiplication $\cdot : \mathbb{K} \times X \mapsto X$ and the map to the norm $\|\| : X \mapsto [0,\infty)$ are continuous.
2. Let X be a $\mathbb{K}$ vector space with inner product $\langle .,. \rangle : X \times X \to \mathbb{K}$. Show that $\langle .,. \rangle$ defines a continuous map from $X \times X$ to $\mathbb{K}$.

Exercise 3

Show that the following maps $T : X \to Y$ define continuous linear operators and determine their operator norm $\|T\|_{X \to Y}$.

1. Let $p \in [1,\infty]$ and $X = Y = l^p$. We define the right shift and left shift operator $T_R x = (0, x_1, x_2, ...)$ and $T_L x = (x_2, x_3, x_4, ...)$, where $x = (x_1, x_2, x_3, ...) \in X$.
2. Let $X = \{f \in C([0,1]) : f(1) = 0\}$ equipped with the supremum norm, $Y = \mathbb{K}$, and define $Tf = \int_0^1 f(s)ds$ for $f \in C([0,1])$.
3. Let $k \in C([0,1]^2; \mathbb{K})$ and $X = Y = C([0,1])$ equipped with the supremum norm. We define $Tf(t) = \int_0^1 k(t,s)f(s)ds$ for $t \in [0,1]$ and $f \in C([0,1])$.

Exercise 4

Show that the vector space $C([-1,1];\mathbb{R})$ together with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$$

for $f, g \in C([-1,1];\mathbb{R})$, is a pre-Hilbert space but not a Hilbert space.