Winter term 2022/2023
Functional Analysis
Exercise sheet 3
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Exercise 1

Let H be a pre-Hilbert space. Show that

1. If $A \subset H$ is a subset, then $A^\perp$ is a closed subspace.
2. If H is a Hilbert space and $B \subset H$ is a closed subspace, then $(B^\perp)^\perp = B$.

Exercise 2

Let $X = C([-1,1];\mathbb{R})$ be equipped with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$. Define the Legendre polynomials $P_n = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$ for $n \in \mathbb{N}_0$. Show that

1. P_n is the unique polynomial of degree n such that $P_n(1) = 1$ and $\int_{-1}^{1} x^m P_n(x)dx = 0$ for all $0 \le m < n$.
2. $(\sqrt{\frac{2n+1}{2}}P_n)_{n\in\mathbb{N}_0}$ is an orthonormal system in $(X, \langle\cdot,\cdot\rangle)$.

Exercise 3

1. Show that $l^p(\mathbb{N})$ is separable for $1 \le p < \infty$, while not separable for $p = \infty$.
2. Show that $l^2(\mathbb{R})$ equipped with the inner product $\langle f, g\rangle = \sum_{x\in\mathbb{R}} f(x)\overline{g(x)}$ is not separable.

Exercise 4

Let H be a Hilbert space and $(x_j)_{j\in\mathbb{N}} \subset H$ an orthonormal set. Show that $(x_j)_{j\in\mathbb{N}}$ is an orthonormal basis if and only if Parseval's identity holds.

Exercise 5

Let H_1 and H_2 be two Hilbert spaces and $T \in L(H_1, H_2)$. Show that

1. There exists a unique operator $T^* \in L(H_2, H_1)$ such that

   $$\langle Tx, y\rangle_{H_2} = \langle x, T^*y\rangle_{H_1}.$$

   for all $x \in H_1$, $y \in H_2$.
2. The operator T^* satisfies $\|T\|_{H_1\mapsto H_2} = \|T^*\|_{H_2\mapsto H_1}$.