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Exercise 1

Prove the Lax-Milgram Lemma: Let H be a Hilbert space and $Q : H \times H \to \mathbb{K}$ such that $Q(\cdot, y)$ is linear and $Q(x, \cdot)$ is antilinear for all $x, y \in H$. Moreover, assume that there exist constants $C, \delta > 0$ such that

$$|Q(x,y)| \leq C\|x\|\|y\|$$
$$Q(x,x) \geq \delta\|x\|^2$$

for all $x, y \in H$. Show that there exists a unique continuous linear map $A : H \to H$ with continuous inverse A^{-1} satisfying

$$Q(x,y) = \langle Ax, y\rangle$$

for all $x, y \in H$. Moreover , show that

$$\|A\|_{H\to H} \leq C, \qquad \|A^{-1}\|_{H\to H} \leq \delta^{-1}.$$

Exercise 2

1. Let $T_L : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be the left shift operator from Exercise 3 of Exercise sheet 2. Determine the adjoint T_L^* and T_L^{**}.
2. Define $T : L^2([0,1]) \to L^2([0,1])$, $Tf(t) = \int_0^t f(s)ds$ for $f \in L^2([0,1])$ and almost every $t \in [0,1]$. Show that $T \in L(L^2([0,1]))$ and determine T^*.
3. Let $k \in L^2([0,1]^2; \mathbb{C})$, and $T_k : L^2([0,1]; \mathbb{R}) \to L^2([0,1]; \mathbb{C})$ be given by

$$T_k f(t) = \int_0^1 k(t,s) f(s) ds$$

for $f \in L^2([0,1]; \mathbb{R})$ and almost every $t \in [0,1]$. Determine T_k^* and give a sharp condition on k such that $T_k = T_k^*$.

Exercise 3

Let $n \in \mathbb{N}$, $0 < p, q \leq \infty$ and let m^n denote the Lebesgue measure. For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$ we define its distribution function $d_f(\alpha) = m^n(\{x \in \mathbb{R}^n : |f(x)| > \alpha\})$, $\alpha > 0$, and the rearrangement $f^*(s) = \inf\{\alpha > 0 : d_f(\alpha) \leq s\}$, $s > 0$. The Lorentz spaces $L^{p,q}$ are defined by

$$L^{p,q} = \{f : \mathbb{R}^n \to \mathbb{C} \text{ measurable } : \|f\|_{L^{p,q}} := \|s^{\frac{1}{p}-\frac{1}{q}} f^*\|_{L^q(0,\infty)} < \infty\}/\sim,$$

where we set $\frac{1}{\infty} := 0$ and $\inf \emptyset := \infty$.

1. Show that $\|\cdot\|_{L^{p,q}}$ is a quasi-norm, i.e., it is positive definite, $\|\lambda f\|_{L^{p,q}} = |\lambda|\|f\|_{L^{p,q}}$ for all $f \in L^{p,q}$, $\lambda \in \mathbb{C}$, and there exists a $c > 0$ such that

$$\|f+g\|_{L^{p,q}} \leq c(\|f\|_{L^{p,q}} + \|g\|_{L^{p,q}})$$

for all $f, g \in L^{p,q}$.

2. Show that $L^p = L^{p,p}$ for all $1 \leq p \leq \infty$.

3. Show that $L^{p,q_1} \subset L^{p,q_2}$ whenever $1 \leq p \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$.

4. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and let $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$, $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$. Show that there exists a constant $C > 0$ such that

$$\|fg\|_{L^{p,q}} \leq C\|f\|_{L^{p_0,q_0}}\|g\|_{L^{p_1,q_1}}$$

for all $f \in L^{p_0,q_0}$ and $g \in L^{p_1,q_1}$. *Hint:* First prove that $(fg)^*(s_1 + s_2) \leq f^*(s_1)g^*(s_2)$ for all $s_1, s_2 > 0$ and measurable functions $f, g : \mathbb{R}^n \to \mathbb{C}$.