Winter term 2022/2023 **Karlsruhe Institute of Technology**
Functional Analysis JProf. Dr. Xian Liao
Exercise sheet 5 M.Sc. Rebekka Zimmermann
Due date: 30.11.2022

Exercise 1

Let $1 \le p < q \le \infty$.

1. Show that $l^p(\mathbb{N}) \subset l^q(\mathbb{N})$, with a strict inclusion.
2. Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that if $\mu(X) < \infty$, then $L^q(\mu) \subset L^p(\mu)$ continuously, i.e., there exists a $C > 0$ such that
$$\|f\|_p \le C\|f\|_q$$
for all $f \in L^q(\mu)$.
3. Show that both $L^p(\mathbb{R}) \not\subseteq L^q(\mathbb{R})$ and $L^q(\mathbb{R}) \not\subseteq L^p(\mathbb{R})$.

Exercise 2

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be integrable.

1. Show that
$$\lim_{n\to\infty} \int_X f\chi_{\{|f|\ge n\}} d\mu = 0,$$
where $\chi_{\{|f|\ge n\}}$ denotes the characteristic function of the set $\{|f| \ge n\} = \{x \in X : |f(x)| \ge n\}$.
2. Show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $A \in \mathcal{A}$ satisfies $\mu(A) < \delta$, then it follows that $\int_A |f| d\mu < \epsilon$.

Exercise 3

Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. Let $(f_n)_{n\in\mathbb{N}} \subset L^1(\mu)$ be a sequence satisfying
$$\sum_{n\in\mathbb{N}} \int_X |f_n| d\mu < \infty.$$
Show that the series $\sum_{n\in\mathbb{N}} f_n(x)$ converges for μ- almost every $x \in X$, that the sum is integrable and
$$\int_X \sum_{n\in\mathbb{N}} f_n d\mu = \sum_{n\in\mathbb{N}} \int_X f_n d\mu.$$
2. Let $f : X \to \mathbb{R}$ be a measurable function. Show that $f \in L^1(\mu)$ if and only if
$$\sum_{n\in\mathbb{Z}} 2^n \mu(\{x \in X : 2^n \le |f(x)| < 2^{n+1}\}) < \infty.$$

Exercise 4

Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \leq p < \infty$. Let $(f_n)_{n\in\mathbb{N}} \subset L^p(\mu)$ be a sequence converging to some $f \in L^p(\mu)$ μ-almost everywhere on X. Show that $f_n \to f$ in $L^p(\mu)$ if and only if $\|f_n\|_{L^p} \to \|f\|_{L^p}$ as $n \to \infty$. *Hint:* First show that $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $a, b \in \mathbb{R}$ and $p \geq 1$, and consider the sequence $g_n = 2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p$ for $n \in \mathbb{N}$.