Winter term 2022/2023 **Karlsruhe Institute of Technology**

Functional Analysis
Exercise sheet 6
Due date: 07.12.2022

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Exercise 1

Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. Let $(Y, \mathcal{B}, \nu)$ be another measure space and let $f : X \to Y$ and $g : Y \to \mathbb{R}$ be measurable, i.e., $f^{-1}(B) \in \mathcal{A}$ for any $B \in \mathcal{B}$. Show that the composition $g \circ f : X \to \mathbb{R}$ is measurable.

2. Let $f, g : X \to \mathbb{R}$ be measurable functions. Show that $f + g$ is measurable.

3. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : X \to \mathbb{R}$ converging μ-almost everywhere to some $f : X \to \mathbb{R}$. Show that f is measurable.

4. Let $f : X \to [0, \infty]$ be measurable. Show that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions $f_n : X \to [0, \infty]$ such that $f_n \le f_{n+1}$ on X for all $n \in \mathbb{N}$ and $f_n \to f$ μ-almost everywhere on X as $n \to \infty$. Here we call a function $g : X \to \mathbb{R}$ a simple function if $g = \sum_{k=1}^{m} a_k \chi_{A_k}$ for some measurable sets $A_k \in \mathcal{A}$ and $a_k \in \mathbb{R}$.

5. Let $f, g \in L^1(\mu)$. Show that
$$\int_X f + g d\mu = \int_X f d\mu + \int_X g d\mu.$$

Exercise 2

Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. Show that if $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then the map $j : L^q(\mu) \to (L^p(\mu))^*$ given by
$$j(g) = (f \mapsto \int_X f g d\mu)$$
for $g \in L^q(\mu)$, is an antilinear isometric isomorphism.

2. Suppose that $(X, \mathcal{A}, \mu)$ is σ finite. Show that the map $j : L^\infty(\mu) \to (L^1(\mu))^*$ given by
$$j(g) = (f \mapsto \int_X f g d\mu)$$
for $g \in L^\infty(\mu)$, is an antilinear isometric isomorphism.