Winter term 2022/2023 **Karlsruhe Institute of Technology**
Functional Analysis JProf. Dr. Xian Liao
Exercise sheet 7 M.Sc. Rebekka Zimmermann
Due date: 14.12.2022

Exercise 1

Let $1 \leq r < \infty$ and $1 < p, q < \infty$ satisfy $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$. The goal of this exercise is to prove Young's inequality in the Lorentz space $L^{q,\infty}$, i.e., that there exists a constant $C > 0$ such that

$$\|f * g\|_{L^{q,\infty}} \leq C\|f\|_{L^r}\|g\|_{L^{p,\infty}}$$

for any $f \in L^r(\mathbb{R}^n)$ and $g \in L^{p,\infty}$.

1. Let $1 \leq \nu < \infty$. Recall that

   $$\|f\|_{L^{\nu,\infty}} = \|s^{\frac{1}{\nu}} f^*(s)\|_{L^\infty((0,\infty))},$$

   defines a quasi-norm on $L^{\nu,\infty}$, where $f^*(s) = \inf\{\alpha > 0 : d_f(\alpha) \leq s\}$ for $s > 0$, and $d_f(\alpha) = m^n(\{x \in \mathbb{R}^n : |f(x)| > \alpha\})$ for $\alpha \geq 0$. Show that

   $$\|f\|_{L^{\nu,\infty}} = \sup_{\alpha > 0} (\alpha^\nu d_f(\alpha))^{\frac{1}{\nu}}$$

   for $f \in L^{\nu,\infty}$.

2. For $g \in L^{p,\infty}$ and $\lambda > 0$ we write $g = g_\lambda + g^\lambda$, where $g_\lambda = g\chi_{\{|g| \leq \lambda\}}$ and $g^\lambda = g\chi_{\{|g| > \lambda\}}$. Show that $g_\lambda \in L^{p_1}(\mathbb{R}^n)$ for $p < p_1 \leq \infty$, and $g^\lambda \in L^{p_2}(\mathbb{R}^n)$ for $1 \leq p_2 < p$.

3. Let $f \in L^r(\mathbb{R}^n)$ and $g \in L^{p,\infty}(\mathbb{R}^n)$. Show that for any $\alpha > 0$ there exists $\lambda > 0$ such that $d_{f*g_\lambda}(\frac{\alpha}{2}) = 0$ and $d_{f*g^\lambda}(\frac{\alpha}{2}) \leq C_{p,q,r}\alpha^{-q}\|f\|_{L^r}^q\|g\|_{L^{p,\infty}}^q$ for some constant $C_{p,q,r} > 0$, and conclude.

Exercise 2

Let $(X, \mathcal{A}, \mu)$ be a measure space, $1 \leq p < \infty$ and let $f, f_n \in L^p(\mu)$ for $n \in \mathbb{N}$.

1. Give an example for f and $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ μ-almost everywhere on X but not $f_n \to f$ in $L^p(\mu)$.

2. Give an example for f and $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ in $L^p(\mu)$ but not $f_n \to f$ μ-almost everywhere on X.

3. Show that if $f_n \to f$ μ-almost everywhere on X and there exists a function $g \in L^p(\mu)$ such that $|f_n| \leq g$ μ-almost everywhere on X then $f_n \to f$ in $L^p(\mu)$.

4. Show that if $f_n \to f$ in $L^p(\mu)$, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ which converges μ-almost everywhere to f.

Exercise 3

Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \le p < \infty$.

1. Show that the set of simple functions $g = \sum_{k=1}^{m} a_k \chi_{A_k}$ such that $A_k \in \mathcal{A}$, $\mu(A_k) < \infty$ and $a_k \in \mathbb{R}$, is dense in $L^p(\mu)$ while not dense in $L^\infty(\mu)$ if $\mu(X) = \infty$.

2. Show that the set $\{(x_j)_{j\in\mathbb{N}} \in l^p(\mathbb{N}) : \exists J \in \mathbb{N}\ \forall j \ge J\ x_j = 0\}$ is dense in $l^p(\mathbb{N})$ while not dense in $l^\infty(\mathbb{N})$.