Exercise 1

1. Prove the Three lines Inequality: Let $\mathcal{C} = \{z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\}$ and suppose that $u \in C(\mathcal{C})$ is bounded and holomorphic in the interior. Then

$$\sup_{\mathcal{C}} |u| = \sup_{\partial \mathcal{C}} |u|.$$

More precisely, for $\theta \in (0,1)$ we have $C_\theta \le C_0^{1-\theta} C_1^\theta$, where $C_\theta = \sup_{t\in\mathbb{R}} |u(\theta + it)|$.

2. Prove the Riesz-Thorin interpolation Theorem: Let $(X, \mathcal{A}, \mu)$ be a σ finite measure space, $1 \le p_0, p_1, q_0, q_1 \le \infty$ and let $T : L^{p_0}(\mu) \cap L^{p_1}(\mu) \to L^{q_0}(\mu) \cap L^{q_1}(\mu)$ be linear satisfying

$$\|Tf\|_{L^{q_0}} \le C_0 \|f\|_{L^{p_0}}, \qquad \|Tf\|_{L^{q_1}} \le C_1 \|f\|_{L^{p_1}}$$

for all $f \in L^{p_0}(\mu) \cap L^{p_1}(\mu)$. Furthermore, let $\theta \in (0,1)$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T defines a unique continuous linear map $L^p(\mu) \to L^q(\mu)$ with

$$\|T\|_{L^p \to L^q} \le C_0^{1-\theta} C_1^\theta.$$

Hint: For simple functions $f = \sum_{k=1}^n \alpha_k \chi_{A_k}$, $g = \sum_{l=1}^m \beta_l \chi_{B_l}$ consider the function $u : \mathcal{C} \to \mathbb{C}$, $u(z) = \int_X g_z T(f_z) d\mu$, where $f_z = \sum_{k=1}^n |\alpha_k|^{a(z)} \frac{\alpha_k}{|\alpha_k|} \chi_{A_k}$, $g_z = \sum_{l=1}^m |\beta_l|^{b(z)} \frac{\beta_l}{|\beta_l|} \chi_{B_l}$ for some suitable functions $a(z)$ and $b(z)$.

Exercise 2

Let (X, d) be a metric space and $K \subset X$ a subset. Show that the following are equivalent.

(i) K is compact, i.e., if $\mathcal{U}$ is a collection of open sets in X such that $K \subset \cup_{U\in\mathcal{U}} U$, then there exist $n \in \mathbb{N}$ and $U_1, ..., U_n \in \mathcal{U}$ such that $K \subset \cup_{i=1}^n U_i$.

(ii) K is sequentially compact, i.e., for any sequence $(x_n)_{n\in\mathbb{N}} \subset K$ there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$.

(iii) K is totally bounded and complete, i.e., for any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $x_1, ..., x_N \in X$ such that $K \subset \cup_{j=1}^N B(x_j, \varepsilon)$, and any Cauchy sequence in K has a subsequence which converges to a limit in K.

Exercise 3

Let (X, d) be a metric space.

1. Let μ be a Borel measure on (X, d). Show that for any Borel set B with $\mu(B) < \infty$ and any $\varepsilon > 0$, there exists a closed set $C \subset B$ with $\mu(B \backslash C) < \varepsilon$.

2. Show that if (X, d) is σ compact there exists a sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of X such that $K_j \subset \mathring{K}_{j+1}$ for all $j \in \mathbb{N}$, and $X = \bigcup_{j=1}^\infty K_j$.