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Exercise 1

For $\alpha \in (0,1)$ we define the space of *α-Hölder continuous functions* on $[0,1]$ as

$$C^\alpha([0,1]) := \{f \in C([0,1]) \,|\, \|f\|_{C^\alpha} := \|f\|_{C_b} + [f]_\alpha < \infty\},$$

where

$$[f]_\alpha := \sup_{\substack{x,y\in[0,1] \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

1. Show that the unit ball $B^\alpha := \{f \in C^\alpha([0,1]) : \|f\|_{C^\alpha} < 1\}$ in $C^\alpha([0,1])$ is relatively compact in $(C([0,1]), \|\cdot\|_{C_b})$, i.e., its closure with respect to $\|\cdot\|_{C_b([0,1])}$ is a compact subset of $C([0,1])$.
2. Show that for $0 < \alpha \leq \beta < 1$ one has the inclusion $C^\beta([0,1]) \subset C^\alpha([0,1])$ and determine all $0 < \alpha \leq \beta < 1$ such that the unit ball $B^\beta = \{f \in C^\beta([0,1]) : \|f\|_{C^\beta} < 1\}$ is relatively compact in $(C^\alpha([0,1]), \|\cdot\|_{C^\alpha})$.

Exercise 2

1. Give an example of a function $\eta \in C^\infty(\mathbb{R}^d)$ which satisfies $\operatorname{supp} \eta \subset B_1(0)$, $0 \leq \eta \leq 1$ and $\int_{\mathbb{R}^d} \eta dm^d = 1$.
2. Let $\eta \in C^\infty(\mathbb{R}^d)$ satisfy the conditions from above, and let $\eta_r(x) = r^{-d}\eta(r^{-1}x)$ for $r > 0$. Let $p \in [1,\infty)$, $f \in L^p(\mathbb{R}^d)$ and $f_r = \eta_r * f$. Show that $f_r \in C^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$ almost everywhere, $r > 0$, and that for any $\epsilon > 0$ there exists $\delta > 0$ such that

   $$\|f_r - f\|_{L^p} < \epsilon$$

   for all $0 < r < \delta$.

Exercise 3

Let $1 \leq p < \infty$. Recall the Kolmogorov-Riesz compactness characterisation: A closed subset $C \subset L^p(\mathbb{R}^d)$ is compact if and only if:

(i) C is bounded.

(ii) For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $|h| < \delta$ and $f \in C$,

$$\|f(\cdot + h) - f\|_{L^p(\mathbb{R}^d)} < \epsilon.$$

(iii) For every $\epsilon > 0$ there exists $R > 0$ such that for every $f \in C$,

$$\|f\chi_{\mathbb{R}^d \setminus B_R(0)}\|_{L^p(\mathbb{R}^d)} < \epsilon.$$

1. Show that (ii) and (iii) already imply (i).
2. Give examples of not relatively compact sets $C \subset L^p(\mathbb{R}^d)$, which do not satisfy (ii) or (iii).

Exercise 4

Let (X, d) be a σ compact metric space, $K \subset X$ a compact set and let $(U_j)_{j=1}^N$ be a finite open covering of K. Show that there exist functions $g_1, ..., g_N : X \to \mathbb{K}$ such that supp $g_j \subset U_j$, $0 \leq g_j \leq 1$ for all $j = 1, ..., N$, and $\sum_{j=1}^N g_j = 1$ on K.