Institute for Analysis
JProf. Dr. Xian Liao

Exam Functional Analysis

28.02.2023

Last name:

First name:

Matriculation number:

Study course:

	Mathematics (Bachelor)
	Mathematics (Master)
	Physics
	Other:

Please write your name and matriculation number on every sheet.

There are **no** auxiliary means allowed.

The exam sheet consists of five problems. For each problem a maximum of **10** points can be achieved. **23** points are required to pass the exam.

You can write your solutions in English or German.

Problem	1	2	3	4	5
Points					

Total points	Mark

Name: Matriculation number: Points:

Problem 1 (3+3+4):
Let $d \in \mathbb{N}$. For $1 \leq p_1, p_2 \leq \infty$ we define

$$L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{R} : f = f_1 + f_2 \text{ a.e. for some } f_1 \in L^{p_1}(\mathbb{R}^d),\ f_2 \in L^{p_2}(\mathbb{R}^d)\},$$

and for $f \in L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d)$ we set

$$\|f\|_{L^{p_1}+L^{p_2}} := \inf\{\|f_1\|_{L^{p_1}} + \|f_2\|_{L^{p_2}} : f_1 \in L^{p_1}(\mathbb{R}^d), f_2 \in L^{p_2}(\mathbb{R}^d) \text{ with } f = f_1 + f_2 \text{ a.e.}\}.$$

1. Show that $L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d)$ is a vector space.
2. Show that $(L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d), \|\cdot\|_{L^{p_1}+L^{p_2}})$ is a normed space.
3. Show that $(L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d), \|\cdot\|_{L^{p_1}+L^{p_2}})$ is a Banach space.
   Hint: You may use without proof that a normed vector space X is complete if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$, there exists an element $x \in X$ such that $\|x - \sum_{n=1}^{N} x_n\|_X \to 0$ as $N \to \infty$.

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Problem 2 (3+3+4):
Let $d \in \mathbb{N}$ and let $U \subset \mathbb{R}^d$ be a bounded C^1-domain. We consider the problem

$$\begin{cases} -\Delta u + u = f, & \text{in } U, \\ u = 0, & \text{on } \partial U, \end{cases} \tag{1}$$

where $f \in L^2(U)$, and $\Delta = \sum_{j=1}^d \partial_{x_j x_j}$ denotes the Laplacian.

1. Show that if $u \in \mathcal{D}(U)(= C_c^\infty(U))$ satisfies (1) and $\varphi \in C^1(\overline{U})$, then there holds
$$\int_U \sum_{j=1}^d (\partial_{x_j} \overline{u} \partial_{x_j} \varphi) + \overline{u}\varphi dx = \int_U \overline{f}\varphi dx.$$
2. Show that $H_0^1(U)$ is a Hilbert space.
3. Show that the map $(H_0^1(U) \ni \varphi \mapsto \int_U \overline{f}\varphi dx)$ is contained in $(H_0^1(U))^*$. Conclude that there exists a unique weak solution $u \in H_0^1(U)$ of (1), i.e., an element $u \in H_0^1(U)$ which satisfies
$$\langle u, \varphi \rangle_{H_0^1(U)} = \langle f, \varphi \rangle_{L^2(U)}$$
for all $\varphi \in H_0^1(U)$.

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Problem 3 (3+4+3):

1. Let $\phi \in L^\infty(\mathbb{R})$ and $p \in (1, \infty)$. For $f \in L^p(\mathbb{R})$ we define $M_\phi f := \phi f$, the multiplication by ϕ. Show that $M_\phi \in L(L^p(\mathbb{R}))$.

2. Let $\chi(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$. Show that $\widehat{e^{-|\cdot|} * f} = 2\chi \hat{f}$ for all $f \in \mathcal{S}(\mathbb{R})$, where $\hat{g}$ denotes the Fourier transform of the function $g \in \mathcal{S}(\mathbb{R})$.

3. For $f \in \mathcal{S}(\mathbb{R})$ we denote $\Lambda(f) := e^{-|\cdot|} * f$. Show that Λ defines an operator in $L(L^2(\mathbb{R}))$.

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Problem 4 (3+3+4):

1. Show that for all $f \in \mathcal{D}(\mathbb{R})(= C_c^\infty(\mathbb{R}))$ one has

$$\|f\|_{C_b(\mathbb{R})} \leq \|f\|_{H^1(\mathbb{R})}.$$

   Hint: Apply the Newton-Leibniz formula (of the form $h(x) = h(a) + \int_a^x h'(y)dy$*, for* $x, a \in \mathbb{R}$*,* $h \in C^1(\mathbb{R})$*) to the function* $f^2 \in \mathcal{D}(\mathbb{R})$*.*

2. Conclude that there exists a continuous embedding $H^1(\mathbb{R}) \to C_b(\mathbb{R})$.

3. Let δ_0 denote the Dirac measure, i.e., $\delta_0(f) = f(0)$ for $f \in C_b(\mathbb{R})$. Show that $\delta_0 \in H^{-1}(\mathbb{R})$.

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Problem 5 (3+3+4):

1. Let $d \in \mathbb{N}$, $(f_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^d)$, and $f, g_j \in L^2(\mathbb{R}^d)$, $j = 1, ..., d$. Suppose that $f_n \rightharpoonup f$ and $\partial_{x_j} f_n \rightharpoonup g_j$ weakly in $L^2(\mathbb{R}^d)$ as $n \to \infty$ for all $j = 1, ..., d$. Show that $f \in H^1(\mathbb{R}^d)$ with $\partial_{x_j} f = g_j$ for $j = 1, ..., d$.

2. Suppose $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are two bounded sequences in $H^1(\mathbb{R}^3)$. Show that there exists a sequence of integers $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $(f_{n_k} g_{n_k})_{k\in\mathbb{N}}$ converges weakly in $L^3(\mathbb{R}^3)$.

3. Give an example of a sequence $(f_n)_{n\in\mathbb{N}} \subset L^6(\mathbb{R}^3)$ and an element $f \in L^6(\mathbb{R}^3)$ such that $f_n \rightharpoonup f$ weakly in $L^6(\mathbb{R}^3)$, but not $f_n^2 \rightharpoonup f^2$ weakly in $L^3(\mathbb{R}^3)$ as $n \to \infty$.
   Hint: Keep in mind the functions $h_n(x_1) = \cos(nx_1)$, *for* $x_1 \in [0, 1]$, $n \in \mathbb{N}$.