Winter term 2022/2023
Functional Analysis
Solution sheet 10
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Karlsruhe Institute of Technology
JProf. Dr. Xian Liao
M.Sc. Rebekka Zimmermann

Exercise 1

Let $X = c_{00} := \{(x_j)_{j\mathbb{N}} \in l^\infty(\mathbb{N}) : \exists J \in \mathbb{N}\ \forall j \geq J\ x_j = 0\}$, $Y = c_0 := \{(x_j)_{j\mathbb{N}} \in l^\infty(\mathbb{N}) : \lim_{j\to\infty} x_j = 0\}$ both endowed with the supremum norm $\|\cdot\|_{l^\infty}$. For $T_n x = (x_1, 2x_2, ..., nx_n, 0, ...)$ for $n \in \mathbb{N}$, $x \in X$, we have $\|T_n\|_{X\to Y} = n$, but $\|T_n x\|_{l^\infty} < \infty$ for all $x \in X$, $n \in \mathbb{N}$.

Exercise 2

For $n \in \mathbb{N}$ we define $O_n = \{f \in C_b(0,1) : \sup_{0<|h|<1/n} \frac{|f(t+h)-f(t)|}{|h|} > n\ \forall t \in [0,1]\}$. If $f \in O_n$ then for every $t \in [0,1]$ there exists $\delta_t > 0$, $0 < |h_t| < \frac{1}{n}$ such that $\frac{|f(t+h_t)-f(t)|}{|h_t|} > n + \delta_t$, and therefore also $\frac{|f(s+h_t)-f(s)|}{|h_t|} > n + \delta_t$ for all $s \in U_t$ for some neighborhood U_t of t. Since $[0,1]$ is compact there exist $t_1, ..., t_k$ such that $[0,1] \subset \cup_{i=1}^k U_{t_i}$, and we set $\delta = \min\{\delta_{t_1}, ..., \delta_{t_k}\}$, $h = \min\{|h_{t_1}|, ..., |h_{t_k}|\}$. Then if $\epsilon \in (0, \frac{\delta}{2}h)$, $g \in C_b(0,1)$ such that $\|f-g\|_{C_b} < \epsilon$, we have $g \in O_n$, and hence O_n is open. Moreover, if $f \in C_b(0,1)$ and $\epsilon > 0$, there exists a polynomial $p \in C_b(0,1)$ such that $\|f-p\|_{C_b} < \frac{\epsilon}{2}$. We choose a function $g_m \in C_b(0,1)$ satisfying $0 \leq g_m \leq \frac{\epsilon}{2}$, $\sup_{0<|h|<1/n} \frac{|g_m(t+h)-g_m(t)|}{|h|} \geq m$ for all $t \in [0,1]$, and set $f_m = p + g_m$. Then $\|f - f_m\|_{C_b} < \epsilon$ and $f_m \in O_n$ for sufficiently large $m \in \mathbb{N}$, from which we deduce that O_n is dense in $C_b(0,1)$ for all $n \in \mathbb{N}$. The Baire Category theorem implies that $\cap_{n\in\mathbb{N}} O_n$ is dense in $C_b(0,1)$.

Exercise 3

T_f is linear and it is continuous by the dominated convergence theorem. The map $L^1_{\text{loc}}(U) \to \mathcal{D}'(U)$, $f \mapsto T_f$ is linear and continuous again by the dominated convergence theorem. If $f \in L^1_{\text{loc}}(U)$ such that $\int_U f\varphi dm^d = 0$ for all $\varphi \in \mathcal{D}(U)$, then for $x_0 \in U$, $R > 0$ such that $B_R(x_0) \subset U$, we have $\int_{\mathbb{R}^d} f\chi\varphi dm^d = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, where $\chi \in \mathcal{D}(\mathbb{R}^d)$ is such that $\operatorname{supp} \chi \subset B_R(x_0)$, $0 \leq \chi \leq 1$, $\chi = 1$ on $B_{R/2}(x_0)$. It follows that $(f\chi) * \varphi(x) = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$. Since there exists a sequence $(\eta_r)_{r>0} \subset \mathcal{D}(\mathbb{R}^d)$ such that $(f\chi) * \eta_r \to f\chi$ in $L^1(\mathbb{R}^d)$ as $r \to 0^+$, this implies that $f = 0$ almost everywhere on $B_{R/2}(x_0)$, and since $x_0 \in U$ was arbitrary we have shown $f = 0$ in $L^1_{\text{loc}}(U)$.

Exercise 4

1. Let $v \in \mathbb{R}^d$, $t \in \mathbb{R}$, $t \neq 0$. Since for any $\psi \in \mathcal{D}(\mathbb{R}^d)$, $\alpha \in \mathbb{N}_0^d$, $t^{-1}((\partial^\alpha\psi)(x+tv) - (\partial^\alpha\psi)(x)) \to \sum_{j=1}^d v_j \partial_j(\partial^\alpha\psi)(x)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^d$, we have as $t \to 0$, $t^{-1}((\phi*T)(x+tv) - (\phi*T)(v)) = T(t^{-1}(\phi(x+tv-\cdot) - \phi(x-\cdot))) \to T(\sum_{j=1}^d v_j(\partial_j\phi)(x-\cdot)) = (\sum_{j=1}^d v_j\partial_j\phi) * T = -T(\sum_{j=1}^d v_j\partial_j(\phi(x-\cdot))) = \phi * (\sum_{j=1}^d v_j\partial_j T)(x)$ for $x \in \mathbb{R}^d$.

2. For $\psi \in L^1(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ we have $\phi * T_\psi(x) = T_\psi(\phi(x-\cdot)) = \int_{\mathbb{R}^d} \psi(y)\phi(x-y)dy = (\psi*\phi)(x)$.

3. Note that if $x \in \mathbb{R}^d$, then $\operatorname{supp} \phi(x-\cdot) = K_1' := \{x-y' : y' \in K_1\}$, and thus if $x \notin K_1 + K_2$ then $x - y' \notin K_2\ \forall y' \in K_1$ which implies $K_2 \cap K_1' = \emptyset$ and hence $T(\phi(x-\cdot)) = 0$.

Exercise 5

1. $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for $p \in [1, \infty)$ by the proof of Theorem 3.34 Step 1. Since one can not approximate the constant one function by $C_c(\mathbb{R}^d)$ functions in $L^\infty(\mathbb{R}^d)$, $C_c(\mathbb{R}^d)$ is not dense in $L^\infty(\mathbb{R}^d)$.

2. If $(f_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R}^d)$ converges to some f with respect to $\|\cdot\|_{C_b}$, and if $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|f - f_n\|_{C_b} < \epsilon$, and there exists $R > 0$ such that $f_n = 0$ on $\mathbb{R}^d \setminus B_R(0)$, hence for all $x \in \mathbb{R}^d \setminus B_R(0)$ we have $|f(x)| \le \|f - f_n\|_{C_b} < \epsilon$, which implies $f \in C_0(\mathbb{R}^d)$. If $f \in C_0(\mathbb{R}^d)$, we take $(\eta_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R}^d)$ satisfying $0 \le \eta_n \le 1$, $\operatorname{supp} \eta_n \subset B_{2n}(0)$, $\eta_n = 1$ on $B_n(0)$ for all $n \in \mathbb{N}$, and define $f_n = f\eta_n$. For given $\epsilon > 0$ we choose $n_0 \in \mathbb{N}$ such that $|f(x)| < \epsilon$ for $x \in \mathbb{R}^d \setminus B_{n_0}(0)$, and then for $n \ge n_0$ we have $\|f - f_n\|_{C_b} < \epsilon$.

3. If $(f_n)_{n\in\mathbb{N}} \subset C_c((0,1))$ converges to some f with respect to $\|\cdot\|_{C_b}$, and if $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\|f - f_n\|_{C_b} < \epsilon$, and there exists $t_0 \in (0,1)$ such that $f_n(t) = 0$ for $t \in [0,1] \setminus (t_0, 1 - t_0)$. It follows that for $t \in [0,1] \setminus (t_0, 1-t_0)$, $|f(t)| < \epsilon$, which implies $f \in C_0([0,1])$. If $f \in C_0([0,1])$, we choose a sequence $(\eta_n)_{n\in\mathbb{N}} \subset C_c((0,1))$ satisfying $0 \le \eta_n \le 1$, $\operatorname{supp} \eta_n \subset B_{1/2-1/(4n)}(\frac{1}{2})$, $\eta_n = 1$ on $B_{1/2-1/(2n)}(\frac{1}{2})$ for all $n \in \mathbb{N}$, and set $f_n = f\eta_n$. For $\epsilon > 0$ there exists $t_0 > 0$ such that for $t \in [0, t_0] \cup [1 - t_0, 1]$, $|f(t)| < \epsilon$, and if $n_0 \in \mathbb{N}$ such that $[t_0, 1 - t_0] \subset B_{1/2-1/(2n)}(\frac{1}{2})$, then for all $n \ge n_0$, $\|f - f_n\|_{C_b} < \epsilon$.