Winter term 2022/2023 **Karlsruhe Institute of Technology**
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Due date: 25.01.2023

Exercise 1

1. Since for $\varphi \in \mathcal{D}((0,1))$ there exists $n_0 \in \mathbb{N}$ such that $\varphi(t) = 0$ for $t \in (0, \frac{1}{n_0})$, we have $|T\varphi| \leq \sum_{n=1}^{n_0} |\varphi^{(n)}(\frac{1}{n})| < \infty$, so that T is well-defined. This also implies that the summation is finite so that T is indeed linear. If $(\varphi_j)_{j\in\mathbb{N}} \subset \mathcal{D}((0,1))$, $\varphi \in \mathcal{D}$ with $\varphi_j \to \varphi$ in $\mathcal{D}((0,1))$, then there exists $n_0 \in \mathbb{N}$ such that $(0, \frac{1}{n_0}] \cap K = \emptyset$, where $K \subset (0,1)$ is compact such that $\operatorname{supp} \varphi_j \subset K$ for all $j \in \mathbb{N}$. Hence $|T\varphi_j - T\varphi| \leq \sum_{n=1}^{n_0} \|\varphi_j^{(n)} - \varphi^{(n)}\|_{C_b} \to 0$ as $j \to \infty$, which shows that T is continuous.

2. If $\chi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp} \chi \subset (-4,4)$, $\chi = 1$ on $(-3,3)$, then $\varphi := \chi e^x \in \mathcal{D}(\mathbb{R})$ and $T\varphi = \sum_{n=1}^\infty e^{1/n} = \infty$, and thus T is not well-defined.

Exercise 2

1. For $m \in \mathbb{N}$ we set $C_m = \{\varphi \in X_K : \sup_{n\in\mathbb{N}} |T_n\varphi| \leq m\}$. Then there exist $m_0 \in \mathbb{N}$, $\varphi_0 \in X_K$, $r > 0$ such that $B_r(\varphi_0) \subset C_{m_0}$. This implies that $B_r(0) \subset C_{2m_0}$. Fix $k_0 \in \mathbb{N}$ such that $2^{-k_0} < r$, and if $\varphi \in X_K$ with $\|\varphi\|_{C_b^{k_0}} \leq 2^{-k_0}$, then $d(\varphi, 0) < r$, from which we deduce that $\sup_{n\in\mathbb{N}} |T_n\varphi| \leq 2m_0$ for $\varphi \in X_K$ such that $\|\varphi\|_{C_b^{k_0}} \leq 2^{-k_0}$. The claim follows with $k = k_0$ and $C = m_0 2^{1+k_0}$.

2. First note that the Arzela-Ascoli theorem implies the compactness of the closed ball $\overline{B_1(0)} = \{f \in X_K : \|f\|_{C_b^{k+1}(K)} \leq 1\}$ in $C_b^k(K)$, so that for $\epsilon > 0$ there exist $f_m \in C_b^k(K)$ $(m = 1, ..., M)$ such that $\overline{B_1(0)} \subset \cup_{m=1}^M \{f \in C_b^k(K) : \|f - f_m\|_{C_b^k} < \epsilon\}$. We may assume that $f_m \in C_c^\infty(U)$. Since there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $m = 1, ..., M$, $|(T_n - T)f_m| \leq \epsilon$, and since for all $f \in \overline{B_1(0)}$ there exists $m \in \{1, ..., M\}$ such that $\|f - f_m\|_{C_b^k} < \epsilon$, we obtain that for all $n \geq n_0$, $|(T_n - T)f| \leq \epsilon(1 + 2C)$, where C is the constant from part 1.

Exercise 3

1. If $x \in \mathbb{R}^d \setminus \{0\}$, $r > 0$ such that $0 \notin B_r(x)$, and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\operatorname{supp} \varphi \subset B_r(x)$, then $\delta_0(\varphi) = \varphi(0) = 0$.

2. For $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we have $(\delta_0 * T)(\varphi) = T(\tilde{\delta_0} * \varphi) = T(\varphi)$, since for $x \in \mathbb{R}^d$, $\tilde{\delta_0} * \varphi(x) = \varphi(x)$.

3. If $\varphi \in \mathcal{D}(\mathbb{R}^d)$, then $|T_{\eta_r}\varphi - \delta_0(\varphi)| = |\int_{B_r(0)} \eta_r(x)(\varphi(x) - \varphi(0))dx| \leq \sup_{|x|<r} |\varphi(x) - \varphi(0)| \to 0$ as $r \to 0^+$. Moreover for $T \in \mathcal{D}'(\mathbb{R}^d)$, $|(\eta_r * T)(\varphi) - T\varphi| = |T(\tilde{\eta_r} * \varphi - \tilde{\delta_0} * \varphi)| \to 0$ as $r \to 0^+$ since $\tilde{\eta_r} * \varphi - \tilde{\delta_0} * \varphi \to 0$ in $\mathcal{D}(\mathbb{R}^d)$ as $r \to 0^+$.

Exercise 4

1. Let $f \in C_c(\mathbb{R}^d)$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in \mathbb{R}^d$ with $|x - y| < \delta$. For all $r \in (0, \delta)$ we then have $|\eta_r * f(x) - f(x)| \leq \epsilon$ for all $x \in \mathbb{R}^d$.

We pick $f \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ which is not uniformly continuous, but $\|\eta_r * f - f\|_{C_b} \to$ as $r \to 0^+$. If $\epsilon > 0$, there exists $r > 0$ such that $\|f - f * \eta_r\|_{C_b} < \frac{\epsilon}{3}$. Note that $\eta_r * f$ is uniformly continuous, and thus we can choose $\delta > 0$ such that $|\eta_r * f(x) - \eta_r * f(y)| < \frac{\epsilon}{3}$ for all $x, y \in \mathbb{R}^d$, $|x - y| < \delta$. In total we obtain $|f(x) - f(y)| < \epsilon$, which implies that f is uniformly continuous, but this is a contradiction.

2. If $f \in C_c^k(U)$ there exists a compact set $K' \subset U$ such that supp $f \subset K'$. Note that there exists $\epsilon > 0$ such that $K := \{x \in \mathbb{R}^d : dist(x, K') \leq \epsilon\} \subset U$, and therefore supp $(\eta_r * f) \subset K$ whenever $r \in (0, \epsilon]$. In order to show $\eta_r * f \to f$ in $C_b^k(U)$ as $r \to 0^+$ it suffices to show $\eta_r * f \to f$ in $C_b(U)$ as $r \to 0^+$, but this was already shown in part 1.

Exercise 5

If $T \in \mathcal{D}'(U)$ has compact support in U, then we can define $S \in \mathcal{D}'(\mathbb{R}^d)$ with the property that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with supp $\varphi \subset U$ we have $T\varphi = S\varphi$. To this end let $\chi \in C_c^\infty(\mathbb{R}^d)$ satisfy supp $\chi \subset U$, $\chi = 1$ on supp T, and for $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we set $S\varphi := T(\chi\varphi)$. Indeed, if $\varphi \in \mathcal{D}(U)$, then $T\varphi = T(\chi\varphi) + T((1-\chi)\varphi) = T(\chi\varphi)$, since supp $T \cap$ supp $((1-\chi)\varphi) = \emptyset$. By Exercise 3.3 it holds that $\eta_r * S \to S$ in $\mathcal{D}(\mathbb{R}^d)$ as $r \to 0^+$, from which we infer that for all $\varphi \in \mathcal{D}(U)$, $(\eta_r * T)\varphi = T(\tilde{\eta_r} * \varphi) = T(\chi(\tilde{\eta_r} * \varphi)) = S(\tilde{\eta_r} * \varphi) = (\eta_r * S)\varphi \to S\varphi = T\varphi$ as $r \to 0^+$, where the inequalities hold for sufficiently small $r > 0$ such that supp $(\tilde{\eta_r} * \varphi) \subset U$. (Note: we can consider $\varphi \in \mathcal{D}(U)$ as a function in $\mathcal{D}(\mathbb{R}^d)$ by extending it by zero outside of U). For all $r > 0$ and $j = 1, ..., d$ we have $\partial_j(\eta_r * T) = \eta_r * (\partial_j T) = 0$, from which it follows that $\eta_r * T = c$ for all $r > 0$ for some constant $c \in \mathbb{R}$, and by taking the limit $r \to 0^+$, we obtain $T = T_c$. For general $T \in \mathcal{D}'(U)$ (not necessarily with compact support) we know there exists $(T_n)_{n\in\mathbb{N}} \subset \mathcal{D}'(U)$ such that T_n has compact support for all $n \in \mathbb{N}$ and $T_n \to T$ in $\mathcal{D}'(U)$ as $n \to \infty$. Then also $\partial_j T_n \to \partial_j T$ as $n \to \infty$ for all $j = 1, ..., d$, so that for sufficiently large n_0, $\partial_j T_n = 0$ for all $n \geq n_0$, $j = 1, ..., d$. The previous step yields $T_n = T_c$ for all $n \geq n_0$ for some $c \in \mathbb{R}$, and hence also $T = T_c$.