Winter term 2022/2023 **Karlsruhe Institute of Technology**
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Exercise 1

1. For $m \in \mathbb{N}$ we set $C_m = \{\varphi \in \mathcal{S}(\mathbb{R}^d) : |T\varphi| \le m\}$. Then there exist $m_0 \in \mathbb{N}$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$, $r > 0$ such that $B_r(\varphi_0) \subset C_{m_0}$, and then also $B_r(0) \subset C_{2m_0}$. If $k_0 \in \mathbb{N}$ such that $2^{-k_0} < r$, and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\sup_{|\alpha|+|\beta|\le k_0} \|x^\alpha \partial^\beta \varphi\|_{sup} \le 2^{-k_0}$, then $|T\varphi| \le 2m_0$, and hence the claim follows with $k = k_0$, $C = m_0 2^{1+k_0}$.

2. The first assertion is proven as in 1. First note that the ball $\overline{B_1(0)} := \{\varphi \in \mathcal{S}(\mathbb{R}^d) : \sup_{|\alpha|+|\beta|\le l+2} \|x^\alpha \partial^\beta \varphi\|_{sup} \le 1\}$ is compact in $S_l := \{\varphi \in C_b^l(\mathbb{R}^d) : \sup_{|\alpha|+|\beta|\le l} \|x^\alpha \partial^\beta \varphi\|_{sup} < \infty\}$. Hence for $\epsilon > 0$ there exist $\varphi_1, ..., \varphi_M \in S_l$ such that $\overline{B_1(0)} \subset \cup_{m=1,...,M}\{\varphi \in S_l : \|\varphi - \varphi_m\|_{S_l} < \epsilon\}$. We may assume that $\varphi_1, ..., \varphi_M \in \mathcal{S}(\mathbb{R}^d)$. If $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and $m = 1, ..., M$, $|(T_n - T)\varphi_m| < \epsilon$, then also $|(T_n - T)\varphi| < (1 + 2C)\epsilon$ for all $\varphi \in \overline{B_1(0)}$.

Exercise 2

1. If $f \in \mathcal{D}(\mathbb{R}^d)$ and $R > 0$ such that supp $f \subset B_R(0)$, then for all $k \in \mathbb{N}$, $\sup_{|\alpha|+|\beta|\le k} \|x^\alpha \partial^\beta f\|_{sup} \le R^{dk} \sup_{|\beta|\le k} \sup_{B_R(0)} |\partial^\beta f| < \infty$. To show that the inclusion $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ is dense, let $f \in \mathcal{S}(\mathbb{R}^d)$ and $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \le \chi \le 1$, supp $\chi \subset B_2(0)$, $\chi = 1$ on $B_1(0)$, then set $\chi_n = \chi(\frac{1}{n}\cdot)$ for $n \in \mathbb{N}$, and define $f_n = f\chi_n$. Then $d(f, f_n) \to 0$ as $n \to \infty$.

2. The inclusion for $p = \infty$ is clear. Let $p < \infty$, $f \in \mathcal{S}(\mathbb{R}^d)$ and $k \in \mathbb{N}$, $k > d$, then $\|f\|_{L^p}^p \le m^d(B_1(0)) \sup_{B_1(0)} |f|^p + \||x|^{-k}\|^p_{\mathbb{R}^d \setminus B_1(0)} \sup_{\mathbb{R}^d \setminus B_1(0)} (|x|^k |f|)^p \le C \sup_{\mathbb{R}^d} (1 + |x|^k)^p |f|^p < \infty$. The density follows from the fact that $\mathcal{D}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.

3. $L^p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ follows from the fact that for all $g \in L^p(\mathbb{R}^d)$ the map $T_g := (\mathcal{S}(\mathbb{R}^d) \ni f \mapsto \int_{\mathbb{R}^d} fg dx)$ is contained in $\mathcal{S}'(\mathbb{R}^d)$. For the denity if $p < \infty$ we note that $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{S}'(\mathbb{R}^d)$.

4. $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$ is clear since $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$, and the density follows from $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$.

Exercise 3

1. Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$. Integrating by parts yields $\mathcal{F}((\frac{1}{i}\partial_x)^\alpha f)(\xi) = (2\pi)^{d/2}(-1)^{|\alpha|} \cdot (-i)^{|\alpha|} \int_{\mathbb{R}^d} f(x)(-i\xi)^\alpha e^{-ix\cdot\xi} dx = \xi^\alpha \mathcal{F}(f)(\xi)$ for all $\xi \in \mathbb{R}^d$. It suffices to show the second identity for $\alpha = e_j$ (j=1,...,d). Note that for all $x, \xi \in \mathbb{R}^d$, $h \neq 0$, $h^{-1}(e^{-ix\cdot(\xi+he_j)} - e^{-ix\cdot\xi}) - (-ix_j)e^{-ix\cdot\xi} \to 0$ as $h \to 0$, and $|h^{-1}(e^{-ix\cdot(\xi+he_j)} - e^{-ix\cdot\xi}) - (-ix_j)e^{-ix\cdot\xi}| = |h^{-1} \int_0^h (-ix_j)(e^{-ix\cdot(\xi+sx_j)} - e^{-ix\cdot\xi}) ds| \le |x_j||e^{-ix\cdot\xi}| \sup_{s\in(0,h)} |e^{-isx_j} - 1| \le 2|x_j|$, so that by dominated convergence it follows that $|h^{-1}(\mathcal{F}(f)(\xi+he_j) - \mathcal{F}(f)(\xi)) - \mathcal{F}(-ix_j f)(\xi) \to 0$ as $h \to 0$.

2. Let $k \in \mathbb{N}$. We are going to show the existence of $N \in \mathbb{N}$, $C > 0$ such that for all α, β with $|\alpha| + |\beta| \le k$ we have $\sup_{\xi\in\mathbb{R}^d} |\xi^\alpha \partial_\xi^\beta \mathcal{F}(f)(\xi)| \le C\|f\|_{N,\mathcal{S}}$. For all $\xi \in$

$\mathbb{R}^d$ we have $|\xi^\alpha \partial_\xi^\beta \mathcal{F}(f)(\xi)| = |\mathcal{F}((-i\partial_x)^\alpha(-ix)^\beta f)(\xi)| \leq (2\pi)^{-d/2}\|\partial_x^\alpha(x^\beta f)\|_{L^1} \leq \|(1+|x|)^{-d-1}\|_{L^1}\|(1+|x|)^{d+1}\partial_x^\alpha(x^\beta f)\|_{L^\infty} \leq C\|f\|_{k+d+1,\mathcal{S}}$.

Exercise 4

1. We are going to show $\int_{\mathbb{R}^d}\mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)}d\xi = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx$ for all $f, g \in \mathcal{S}(\mathbb{R}^d)$, which then implies $\|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2}$ for all $f \in \mathcal{S}(\mathbb{R}^d)$ so that we can extend $\mathcal{F}$ onto $L^2(\mathbb{R}^d)$ by the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$. To this end we claim that for all $\epsilon > 0$ and $x \in \mathbb{R}^d$, $\mathcal{F}(e^{-\frac{\epsilon^2}{2}|\xi|^2})(x) = \epsilon^{-d}e^{-\frac{1}{2\epsilon^2}|x|^2}$. If we assume the claim to be true, then we have for all $f, g \in \mathcal{S}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d}\mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)}d\xi = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}e^{iy\cdot\xi}\overline{g(y)}dyd\xi f(x)dx$, where for all $x \in \mathbb{R}^d$, $(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}e^{iy\cdot\xi}\overline{g(y)}dyd\xi = \lim_{\epsilon\to 0}(2\pi)^{-d}\int_{\mathbb{R}^d} e^{-\frac{\epsilon^2}{2}|\xi|^2}e^{ix\cdot\xi}\int_{\mathbb{R}^d} e^{-iy\cdot\xi}\overline{g(y)}dyd\xi = \lim_{\epsilon\to 0}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{\epsilon^2}{2}|\xi|^2}e^{-i(y-x)\cdot\xi}d\xi\overline{g(y)}dy = \lim_{\epsilon\to 0}(2\pi)^{-d/2}\int_{\mathbb{R}^d}\mathcal{F}(e^{-\frac{\epsilon^2}{2}|\xi|^2})(y-x)\overline{g(y)} = \lim_{\epsilon\to 0}(2\pi)^{-d/2}\int_{\mathbb{R}^d}\epsilon^{-d}e^{-\frac{1}{2}|\frac{y-x}{\epsilon}|^2}\overline{g(y)}dy = \overline{g(x)}$, since $(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}|z|^2}dz = 1$. In order to prove the claim, by substitution it suffices to prove $\mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi) = e^{-\frac{1}{2}|\xi|^2}$ for $\xi \in \mathbb{R}^d$. Note that $(\partial_{x_j} + x_j)e^{-\frac{1}{2}|x|^2} = 0$ for $j = 1, ..., d$, so that by applying $\mathcal{F}$ we obtain $i(\xi_j + \partial_{\xi_j})\mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi) = 0$. If $d = 1$ and $\phi(\xi) := \mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi)$, then $\phi' + \xi\phi = 0$ (note $\phi \in \mathcal{S}(\mathbb{R}^d)$), and thus, $(e^{\frac{1}{2}\xi^2}\phi)' = 0$, so that there exists $C \in \mathbb{R}$ such that $\phi(\xi) = Ce^{-\frac{1}{2}\xi^2}$, where $C = \phi(0) = \mathcal{F}(e^{-\frac{1}{2}x^2})(0) = 1$. Hence we obtain $\mathcal{F}(e^{-\frac{1}{2}x^2}) = \phi(\xi) = e^{-\frac{1}{2}\xi^2}$. For $d \geq 2$ we use induction noticing that $\mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi) = (2\pi)^{-(d-1)/2}\int_{\mathbb{R}^{d-1}} e^{-ix'\cdot\xi'}e^{-\frac{1}{2}|x'|^2}dx'(2\pi)^{-1/2}\int_{\mathbb{R}} e^{-ix_d\xi_d}e^{-\frac{1}{2}x_d^2}dx_d = \mathcal{F}(e^{-\frac{1}{2}|x'|^2})(\xi')e^{-\frac{1}{2}\xi_d^2}$, where $x' = (x_1, ..., x_{d-1})$, $\xi' = (\xi_1, ..., \xi_{d-1})$. In order to show $\mathcal{F}(f)(\xi) = \lim_{R\to\infty}(2\pi)^{-d/2}\int_{B_R(0)} e^{-ix\cdot\xi}f(x)dx$ for $f \in L^2(\mathbb{R}^d)$ and almost every $\xi \in \mathbb{R}^d$, let $R > 0$ and define $f_R = f\chi_{B_R(0)}$. Then $f_R \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for all $R > 0$, so that $\mathcal{F}(f_R)$ is well-defined with $\mathcal{F}(f_R)(\xi) = (2\pi)^{-d/2}\int_{B_R(0)} e^{-ix\cdot\xi}f(x)dx$ for almost every $\xi \in \mathbb{R}^d$. Since $\mathcal{F}$ is unitary on $L^2(\mathbb{R}^d)$ we have $\|\mathcal{F}(f) - \mathcal{F}(f_R)\|_{L^2} = \|f - f_R\|_{L^2} = (\int_{\mathbb{R}^d\setminus B_R(0)}|f(x)|^2dx)^{1/2} \to 0$ as $R \to \infty$, from which we deduce the claimed identity.

2. We have seen $\|\mathcal{F}\|_{L^1\to L^\infty} \leq (2\pi)^{-d/2}$ and $\|\mathcal{F}\|_{L^2\to L^2} = 1$. By the Riesz-Thorin interpolation theorem it follows that for all $\theta \in (0, 1)$ and p, q defined by $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$, $\frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$ we have $\|\mathcal{F}\|_{L^p\to L^q} \leq (2\pi)^{-\frac{d}{2}(1-\theta)} \cdot 1^\theta$. Note that $p = \frac{2}{2-\theta} \in (1, 2)$, $q = \frac{2}{\theta}$, $\frac{1}{p} + \frac{1}{q} = 1$ and $(1-\theta) = \frac{2}{p} - 1$.