Winter term 2022/2023 **Karlsruhe Institute of Technology**
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Exercise 1

1. If for $|\alpha| \leq k$, D^α denotes the strong derivative, then we have by the integration by parts formula, $\int_U (D^\alpha f)\phi dx = (-1)^\alpha \int_U f D^\alpha \phi dx$, which yields the claim.
2. Let $U = (-1,1)$, $f = |\cdot|$. Then its weak derivative is given by $g = -\chi_{(-1,0)} + \chi_{(0,1)}$.
3. The one function $1 \in W^{k,p}(U) \setminus W_0^{k,p}(U)$.
4. Let $d = 2$, $1 \leq p < 2$, $U = \{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, x_1^\gamma < x_2 < 1\}$ with $\gamma \in (0,1)$ satisfying $1 - \frac{p}{2} > \frac{\gamma}{\gamma+1}$. Then $f(x_1,x_2) = x_2^{-\alpha}$, $\alpha = \frac{(\gamma+1)(2-p)}{2\gamma p}$, satisfies $f \in W^{1,p}(U) \setminus L^{\frac{2p}{2-p}}(U)$. Hence by the Sobolev inequality, there can not be a liner continuous extension operator $W^{1,p}(U) \to W^{1,p}(\mathbb{R}^2)$.

Exercise 2

1. Let $(g_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d)$ such that $g_n \to g$ in $C_b^k(U)$ as $n \to \infty$. For all $\phi \in \mathcal{D}(U)$, $j = 1,...,d$, $n \in \mathbb{N}$, we have $\int_U f g_n \partial_{x_j}\phi dx = \int_U f(\partial_{x_j}(\phi g_n) - \phi \partial_{x_j} g_n) dx = -\int_U ((\partial_{x_j} f) g_n + f(\partial_{x_j} g))\phi dx$. By the dominated converge theorem the equality also holds if we let $n \to \infty$. Inductively we obtain $gf \in W^{k,p}(U)$.
2. Let $p < \infty$ and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d)$ such that $f_n \to f$ in $W^{k,p}(U)$ as $n \to \infty$. Then for all $\varphi \in \mathcal{D}(U)$ and $j = 1,...,d$, $\int_V f_n(\phi(x))\partial_{x_j}\varphi(x)dx = -\sum_{k=1}^d \int_V (\partial_{x_k} f_n)(\phi(x))(\partial_{x_j}\phi_k(x))\varphi(x)dx$. By the dominated converges theorem we obtain $\partial_{x_j}(f(\phi)) = \sum_{k=1}^d (\partial_{x_k} f)(\phi)(\partial_{x_j}\phi_k)$. Here we denote $\phi = (\phi_1,...,\phi_d)$. By substitution we have $\|\partial_{x_j}(f(\phi))\|_{L^p(V)} \leq C\sum_{k=1}^d \|\partial_{x_k} f\|_{L^p(U)} \|D\phi\|_{C_b}$, where C depends on $\|D\phi^{-1}\|_{C_b}$. Inductively together with 1 the claim follows. If $p = \infty$ we use that if $\phi \in \mathcal{D}(U)$, supp $\phi \subset \tilde{U}$ with $\tilde{U}$ open and bounded, then $f \in L^q(\tilde{U})$ for all $q \in [1,\infty)$.

Exercise 3

For $\phi \in \mathcal{D}(\mathbb{R})$ and $0 \leq l \leq k$ we compute $(-1)^l \int_\mathbb{R} F(x)\frac{d^l}{dx^l}\phi(x)dx = (-1)^l \int_{-\infty}^0 f(x)\frac{d^l}{dx^l}\phi(x)dx + (-1)^l \sum_{j=1}^{k+1} a_j \int_0^\infty f(-jx)\frac{d^l}{dx^l}\phi(x)dx = (-1)^l \int_{-\infty}^0 f(x)\frac{d^l}{dx^l}\phi(x)dx - \sum_{j=1}^{k+1} a_j(-1)^l(-j)^l(-j)^{-1} \cdot \int_{-\infty}^0 f(x)\frac{d^l}{dx^l}(\phi(-\frac{x}{j}))dx = \int_{-\infty}^0 f(x)\frac{d^l}{dx^l}((-1)^l\phi(x) - \sum_{j=1}^{l+1} a_j j^l(-j)^{-1}\phi(-\frac{x}{j}))dx$. In order to move the derivative $\frac{d^l}{dx^l}$ onto f we need to show that for all $0 \leq l' < l$, $\frac{d^{l'}}{dx^{l'}}((-1)^l\phi(x) - \sum_{j=1}^{l+1} a_j j^l(-j)^{-1}\phi(-\frac{x}{j}))|_{x=0} = 0$, since then the function $(-\infty,0) \ni x \mapsto ((-1)^l\phi(x) - \sum_{j=1}^{l+1} a_j j^l(-j)^{-1}\phi(-\frac{x}{j}))$ can be approximated by $\mathcal{D}((-\infty,0))$ functions and we can use that $f \in W^{k,p}(V)$. By the definition of the a_j we have $\frac{d^{l'}}{dx^{l'}}((-1)^l\phi(x) - \sum_{j=1}^{l+1} a_j j^l(-j)^{-1}\phi(-\frac{x}{j}))|_{x=0} = (-1)^l\frac{d^{l'}}{dx^{l'}}\phi(0) - \sum_{j=1}^{k+1} a_j(-1)^{1+l'} j^{-1-j'}\frac{d^{l'}}{dx^{l'}}\phi(0) = (-1)^{1+l'}\frac{d^{l'}}{dx^{l'}}\phi(0)((-1)^{l-1-l'} - \sum_{j=1}^{k+1} a_j j^{l-1-l'}) = 0$. With this we arrive at $(-1)^l \int_\mathbb{R} F(x)\frac{d^l}{dx^l}\phi(x)dx = \int_{-\infty}^0 (\frac{d^l}{dx^l}f)(x)(\phi(x) - (-1)^l \sum_{j=1}^{k+1} a_j j^{l-1}\phi(-\frac{x}{j})) = \int_\mathbb{R} F_l(x)\phi(x)dx$.

Exercise 4

For almost every $x \in B_R(0)$ we have $f(x) - f_B(x) = \frac{1}{m^d(B_R(0))} \int_{S^{d-1}} \int_0^{R_\varphi} \int_0^r (\nabla f)(x + s\varphi) \cdot \varphi ds r^{d-1} dr d\varphi$, where $S^{d-1} = \{\varphi \in \mathbb{R}^d : |\varphi| = 1\}$ and $R_\varphi = \sup\{r > 0 : x + r\varphi \in B_R(0)\}$. It follows that $|f(x)-f_B(x)| \leq \frac{1}{m^d(B_R(0))} \frac{(2R)^d}{d} \int_{S^{d-1}} \int_0^{R_\varphi} |(\nabla f)(x+s\varphi)| ds d\varphi = \frac{1}{m^d(B_R(0))} \frac{(2R)^d}{d} \int_{B_R(0)} |\nabla f(y)||x-y|^{-(d-1)} dy \leq \frac{1}{m^d(B_R(0))} \frac{(2R)^d}{d} \||\nabla f||x-y|^{-(d-1)/p}\|_{L^p(B_R(0))} \||x-y|^{-(d-1)/p'}\|_{L^p(B_R(0))} \leq C_{R,d} \||\nabla f||x-y|^{-(d-1)/p}\|_{L^p(B_R(0))}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, and thus $\int_{B_R(0)} |f(x) - f_B(x)|^p dx$
$\leq C'_{R,d} \int_{B_R(0)} |\nabla f(y)|^p \int_{B_R(0)} \frac{1}{|x-y|^{d-1}} dx dy \leq C''_{R,d} \|\nabla f\|^p_{L^p(B_R(0))}$.

Exercise 5

Let $f \in \mathcal{D}(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$, $r = \frac{|x-y|}{2}$. We compute $m^d(B_1(0)) r^d |f(x) - f(y)| = \int_{B_r(\frac{x+y}{2})} |f(x) - f(y)| dz \leq \int_{B_r(\frac{x+y}{2})} |f(x) - f(z)| dz + \int_{B_r(\frac{x+y}{2})} |f(y) - f(z)| dz \leq \int_{B_r(\frac{x+y}{2})} \int_0^1 |(\nabla f)(x + t(z-x))||x-z| + |(\nabla f)(y+t(z-y))||y-z| dt dz \leq 2r \int_0^1 \int_{B_{rt}(x+t(\frac{x+y}{2}-x))} t^{-d} |\nabla f(z')| dz' + \int_{B_{rt}(y+t(\frac{x+y}{2}-y))} t^{-d} |\nabla f(z')| dz' dt \leq 2r \|\nabla f\|_{L^p(B_{2r(x)})} \int_0^1 t^{-d} 2(m^d(B_1(0)(rt)^d))^{\frac{p-1}{p}} dt \leq m^d(B_1(0)) r^d |x-y|^{1-\frac{d}{p}} \frac{4p}{p-d} m^d(B_1(0))^{-\frac{1}{p}} \|\nabla f\|_{L^p(B_{2r}(x))}$.
Hence we showed $|f(x) - f(y)| \leq c_{d,p} |x-y|^{1-\frac{d}{p}} \|\nabla f\|_{L^p(B_{|x-y|}(x))}$. For general $\tilde{f} \in W^{1,p}(U)$ let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d)$ such that $f_n \to \tilde{f}$ in $W^{1,p}(U)$ as $n \to \infty$ and $f_n \to \tilde{f}$ almost everywhere on U as $n \to \infty$. Since $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^{1-\frac{d}{p}}(U)$, which is complete, there exists a limit $g \in C^{1-\frac{d}{p}}(U)$. Then almost everywhere on U we have $\tilde{f} = \lim_{n\to\infty} f_n = g$.