Winter term 2022/2023 **Karlsruhe Institute of Technology**
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Exercise 1

1. If $x \in l^p$ such that $\|x\|_{l^p} = 1$, then $|x_j|^q \leq |x_j|^p$ for all $j \in \mathbb{N}$, and hence $\sup_{N\in\mathbb{N}} \sum_{j=1}^N |x_j|^q \leq \sup_{N\in\mathbb{N}} \sum_{j=1}^N |x_j|^p \leq \|x\|_{l^p}^p = 1$. Thus $x \in l^q$ and $\|x\|_{l^q} \leq 1$. For arbitrary $x \in l^p \setminus \{0\}$ we have $\|x\|_{l^q} = \|x\|_{l^p} \| \frac{x}{\|x\|_{l^p}} \|_{l^q} \leq \|x\|_{l^p}$, which implies $l^p \subset l^q$ with a continuous embedding. Let $s \in (\frac{1}{q}, \frac{1}{p})$, then $(j^{-s})_{j\in\mathbb{N}} \in l^q \setminus l^p$.

2. If $q = \infty$ we have $\|f\|_p \leq \|f\|_\infty \mu(X)^{\frac{1}{p}}$ for $f \in L^\infty(\mu)$. If $q < \infty$, then $\|f\|_p^p \leq (\int_X |f|^q d\mu)^{\frac{p}{q}} (\int_X 1 d\mu)^{\frac{q-p}{q}} = \|f\|_q^p (\mu(X))^{\frac{q-p}{q}}$ for $f \in L^q(\mu)$ which implies $\|f\|_p \leq \|f\|_q (\mu(X))^{\frac{q-p}{qp}}$.

3. Let $s \in (-\frac{1}{p}, -\frac{1}{q})$ and set $f(x) = x^s$ if $x \in [0,1]$ and zero otherwise. Then $f \in L^p(\mathbb{R}^n) \setminus L^q(\mathbb{R}^n)$. Let $x \in l^q \setminus l^p$ and define $f = \sum_{j=1}^\infty x_j \chi_{[j,j+1)}$. Then $\|f\|_q = \|x\|_q < \infty$ and $\|f\|_p = \|x\|_p = \infty$.

Exercise 2

1. Since $|f|\chi_{\{|f|\geq n\}} \to 0$ almost everywhere and $|f|\chi_{\{|f|\geq n\}} \leq |f|$ almost everywhere, the claim follows by the dominated convergence theorem.

2. Assume there exists an $\epsilon > 0$ such that for all $n \in \mathbb{N}$ there exists $A_n \in \mathcal{A}$ satisfying $\mu(A_n) < \frac{1}{n}$ such that $\int_{A_n} |f| d\mu \geq \epsilon$. Since by the dominated convergence theorem we have $\int_{A_n} |f| d\mu \to 0$ this is a contradiction.

Exercise 3

1. By Beppo-Levi's monotone convergence theorem we have $\sum_{n=1}^\infty \int_X |f_n| d\mu = \int_X \sum_{n=1}^\infty |f_n|$ which is finite by hypothesis, and hence $\sum_{n=1}^\infty |f_n(x)|$ converges for almost every $x \in X$. By the dominated convergence theorem the desired integral identity follows.

2. We have $\sum_{n\in\mathbb{Z}} 2^n \mu(\{x \in X : 2^n \leq |f(x)| < 2^{n+1}\}) \leq \sum_{n\in\mathbb{Z}} \int_X \chi_{\{x\in X: 2^n \leq |f(x)| < 2^{n+1}\}} |f(x)| d\mu = \int_X |f| d\mu$ and $\sum_{n\in\mathbb{Z}} 2^n \mu(\{x \in X : 2^n \leq |f(x)| < 2^{n+1}\}) \geq \frac{1}{2} \sum_{n\in\mathbb{Z}} \int_X \chi_{\{x\in X: 2^n \leq |f(x)| < 2^{n+1}\}} |f(x)| d\mu = \frac{1}{2} \int_X |f| d\mu$.

Exercise 4

The implication from left to right is clear. Assume that $\|f_n\|_p \to \|f\|_p$. For $a, b \geq 0$ and $p > 1$ the map $a \mapsto (a+b)^p - 2^{p-1}a^p$ attains its maximum at $a = b$ and therefore $(a+b)^p - 2^{p-1}a^p \leq 2^{p-1}b^p$ which implies $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. The inequality also holds for $p = 1$. Applying Fatou's lemma to $g_n = 2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p$ implies $\int_X 2^p |f|^p d\mu \leq \liminf_{n\to\infty} \int_X g_n d\mu \leq \limsup_{n\to\infty} \int_X g_n d\mu$ and therefore $2^p \|f\|_p^p \leq 2^p \|f\|_p^p - \limsup_{n\to\infty} \|f - f_n\|_p^p$ which implies $\|f - f_n\|_p \to 0$.