Winter term 2022/2023 | **Karlsruhe Institute of Technology**

Functional Analysis | JProf. Dr. Xian Liao

Solution sheet 7 | M.Sc. Rebekka Zimmermann

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Exercise 1

1. If we set $C = \sup_{\alpha>0}(\alpha^\nu d_f(\alpha))^{\frac{1}{\nu}}$ then $d_f(\alpha) \leq \frac{C^\nu}{\alpha^\nu}$ for all $\alpha > 0$ and hence $f^*(s) \leq \frac{C}{s^{1/\nu}}$ which implies $\|f\|_{\nu,\infty} \leq C$. Moreover, for $\alpha > 0$, $\epsilon \in (0, \alpha)$ we have $f^*(d_f(\alpha) - \epsilon) > \alpha$ and hence $\sup_{s>0} s^{1/\nu} f^*(s) \leq (d_f(\alpha) - \epsilon)^{1/\nu}\alpha$. By letting $\epsilon \to 0$ we obtain $\|f\|_{\nu,\infty} \geq C$.

2. It is clear that $g_\lambda \in L^\infty$. For $p < p_1 < \infty$ we compute $\int_{\mathbb{R}^n} |g_\lambda|^{p_1} dx = p_1 \int_0^\infty \alpha^{p_1-1} d_{g_\lambda}(\alpha) d\alpha = p_1 \int_0^\lambda \alpha^{p_1-1}(d_g(\alpha) - d_g(\lambda)) d\alpha \leq \frac{p_1}{p_1-p}\lambda^{p_1-p}\|g\|_{p,\infty}^p - \lambda^{p_1} d_g(\lambda) \leq \frac{p_1}{p_1-p}\lambda^{p_1-p}\|g\|_{p,\infty}^p$. Furthermore, for $1 \leq p_2 < p$ we have $\int_{\mathbb{R}^n} |g^\lambda|^{p_2} dx = p_2 \int_0^\infty \alpha^{p_2-1} d_{g^\lambda}(\alpha) d\alpha = p_2 \int_0^\lambda \alpha^{p_2-1} d_g(\lambda) d\alpha + p_2 \int_\lambda^\infty \alpha^{p_2-1} d_g(\alpha) d\alpha \leq \lambda^{p_2-p}\|g\|_{L^{p,\infty}}^p + \frac{p_2}{p-p_2}\lambda^{p_2-p}\|g\|_{L^{p,\infty}}^p = \frac{p}{p-p_2}\lambda^{p_2-p}\|g\|_{L^{p,\infty}}^p$.

3. Let $f \in L^r(\mathbb{R}^n)$, $g \in L^{p,\infty}$ and $\alpha > 0$. Note that $d_{f*g}(\alpha) \leq d_{f*g_\lambda}(\frac{\alpha}{2}) + d_{f*g^\lambda}(\frac{\alpha}{2})$ and $|f * g_\lambda(x)| \leq \|f\|_{L^r}\|g_\lambda\|_{L^{r'}}$ for $x \in \mathbb{R}^n$, where $\frac{1}{r} + \frac{1}{r'} = 1$. If $r' < \infty$, the previous calculation shows $|f * g_\lambda(x)| \leq \|f\|_{L^r}(\frac{r'}{r'-p}\lambda^{r'-p}\|g\|_{L^{p,\infty}}^p)^{\frac{1}{r'}}$, and $|f * g_\lambda(x)| \leq \lambda\|f\|_{L^r}$ if $r' = \infty$. With λ such that $\frac{\alpha}{2} = \|f\|_{L^r}(\frac{r'}{r'-p}\lambda^{r'-p}\|g\|_{L^{p,\infty}}^p)^{\frac{1}{r'}}$ if $r' < \infty$, and $\frac{\alpha}{2} = \lambda\|f\|_{L^r}$ if $r' = \infty$, we obtain $d_{f*g_\lambda(\frac{\alpha}{2})} = 0$. Young's inequality implies $\|f * g^\lambda\|_{L^r} \leq \|f\|_{L^r}\|g^\lambda\|_{L^1}$ and hence, $d_{f*g^\lambda}(\frac{\alpha}{2}) \leq \int_{\mathbb{R}^n}(\frac{2}{\alpha}\|f*g^\lambda\|_{L^r})^r \leq (\frac{2}{\alpha})^r(\frac{p}{p-1})^r\|f\|_{L^r}^r\lambda^{(1-p)r}\|g\|_{p,\infty}^{pr} \leq C_{p,q,r}(\frac{2}{\alpha})^q\|f\|_{L^r}^q\|g\|_{L^{p,\infty}}^q$, where the last inequality follows by inserting λ and noticing that $r - \frac{r'}{r'-p}(1-p)r = q$ if $r' < \infty$. In total we showed $\|f * g\|_{L^{q,\infty}} = \sup_{\alpha>0}(\alpha^q d_{f*g}(\alpha))^{1/q} \leq \sup_{\alpha>0}(\alpha^q(\frac{2}{\alpha})^q C_{p,q,r}\|f\|_{L^r}^q\|g\|_{L^{p,\infty}}^q)^{1/q} = \tilde{C}_{p,q,r}\|f\|_{L^r}\|g\|_{L^{p,\infty}}$ which proves the claim.

Exercise 2

1. We choose $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{M}, m)$, where m denotes the Lebesgue measure, and we set $f_n = n^{1/p}\chi_{[0,\frac{1}{n}]}$. Then $f_n \to 0$ almost everywhere, but $\|f_n\|_{L^p} = 1$ for all $n \in \mathbb{N}$.

2. We again choose $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{M}, m)$ and for $n \in \mathbb{N}$, $n = 2^m + k$, $m \in \mathbb{N}_0$, $k \in \{0, ..., 2^m - 1\}$ we set $f_n(x) = \chi_{[0,1]}(2^m x - k)$. Then $\|f_n\|_{L^p} = 2^{-m/p} \to 0$ as $n \to \infty$ but for all $x \in [0, 1]$ there exist subsequences $(f_{n_j})_{j\in\mathbb{N}}$ and $(f_{n_{j'}})_{j'\in\mathbb{N}}$ such that $f_{n_j}(x) = 0$ and $f_{n_{j'}}(x) = 1$ for all $j, j' \in \mathbb{N}$.

3. Let $h_n = |f_n - f|^p$. Then $h_n \to 0$ and $|h_n| \leq (2g)^p$ almost everywhere which implies $\lim_{n\to\infty}\int_X h_n d\mu = 0$ and hence $f_n \to f$ in $L^p(\mu)$.

4. Let $(f_{n_k})_{k\in\mathbb{N}}$ be a subsequence such that $\|f_{n_i} - f_{n_j}\|_{L^p} < 2^{-k}$ for $i, j \geq k$. Set $g_1 = f_{n_1}$, $g_k = f_{n_k} - f_{n_{k-1}}$, $k \geq 2$. Then $\sum_{k=1}^\infty \|g_k\|_{L^p} \leq \|f_{n_1}\|_{L^p} + \sum_{k=1}^\infty 2^{-k} < \infty$ from which it follows that $h_k := \sum_{j=1}^k |g_j| \in L^p$ with $\|h_k\|_{L^p} \leq \sum_{k=1}^\infty \|g_k\|_{L^p}$. By the monotone convergence theorem, $h := \lim_{k\to\infty} h_k = \sum_{j=1}^\infty |g_j|$ satisfies $\int_X h^p dx \leq (\sum_{k=1}^\infty \|g_k\|_{L^p})^p$ which implies $h < \infty$ almost everywhere and hence $\sum_{k=1}^\infty g_k$ is absolutely convergent almost everywhere on X and is contained in L^p. Moreover, $\|\sum_{j=1}^\infty g_j - \sum_{j=1}^k g_j\|_{L^p} \to 0$ by the dominated convergence theorem since $|\sum_{j=1}^\infty g_j - \sum_{j=1}^k g_j|^p \leq 2^p h^p$ almost everywhere. More precisely, we showed that $f_{n_k} = \sum_{j=1}^k g_j \to \sum_{j=1}^\infty g_j$ almost everywhere on X and in L^p and since also $f_{n_k} \to f$ in L^p it follows that $f = \sum_{j=1}^\infty g_j$ almost everywhere.

Exercise 3

1. Let $f \in L^p(\mu)$. $f \geq 0$, by Exercise Sheet 6 Exercise 1 there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $0 \leq f_n \leq f_{n+1} \leq f$ and $f_n \to f$ almost everywhere on X. For all $n \in \mathbb{N}$, $f_n \in L^p(\mu)$ since $0 \leq f_n \leq f$. Moreover, $|f_n - f|^p \to 0$ and $|f_n - f|^p \leq (2|f|)^p$ almost everywhere. The dominated convergence theorem implies $f_n \to f$ in $L^p(\mu)$. Note that if we choose f_n as in Exercise 1 on Exercise Sheet 6, then $f_n = \sum_{k=1}^{n2^n} 2^{-n}(k-1)\chi_{f^{-1}([2^{-n}(k-1),2^{-n}k))}$, where $\mu(f^{-1}([2^{-n}(k-1), 2^{-n}k))) < \infty$ since $f \in L^p(\mu)$. For arbitrary $f \in L^p(\mu)$ we write $f = f^+ - f^-$ and let $\epsilon > 0$. By the previous argument there exist f_1, f_2 simple functions which vanish outside a set of finite measure such that $\|f^+ - f_1\|_{L^p} < \frac{\epsilon}{2}$, $\|f^- - f_2\|_{L^p} < \frac{\epsilon}{2}$. Note that $f_0 := f_1 - f_2$ is a simple function which vanishes outside a set of finite measure, and $\|f - f_0\|_{L^p} < \epsilon$. In order to show that the set of simple functions which vanish outside a set of finite measure is not dense in $L^\infty(\mu)$ if $\mu(X) = \infty$, we consider the function $f = 1$. Since for any simple function there exists a set $A \in \mathcal{A}$ such that $\mu(A) > 0$ and $g = 0$ on A we deduce that $\|f - g\|_{L^\infty} \geq 1$. Hence, f cannot be approximated by simple function which vanish outside a set of finite measure.

2. We set $A := \{(x_j)_{j\in\mathbb{N}} \in l^p(\mathbb{N}) : \exists J \in \mathbb{N}\ \forall j \geq J\ x_j = 0\}$. Let $(x_j)_{j\in\mathbb{N}} \in l^p(\mathbb{N})$ and $\epsilon > 0$. Since $\sum_{j=1}^{\infty} |x_j|^p < \infty$ there exists $J = J(\epsilon) \in \mathbb{N}$ such that $\sum_{j=J}^{\infty} |x_j|^p < \epsilon^p$. If we set $y_j = x_j$ if $j < J$ and zero otherwise we obtain $y \in A$ and $\|x - y\|_{l^p} < \epsilon$. To show that A is not dense in $l^\infty(\mathbb{N})$ we consider the sequence $x = (1, 1, 1, ...)$. Since for any $y \in A$ there exists $j \in \mathbb{N}$ such that $y_j = 0$ we always have $\|x - y\|_{l^\infty} \geq 1$, it follows that y cannot be approximated by elements in A.