Winter term 2022/2023 **Karlsruhe Institute of Technology**
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Exercise 1

1. If $(f_n)_{n\in\mathbb{N}} \subset B^\alpha$ then $\|f_n\|_{C_b} < 1$ which implies that $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C([0,1])$, and $[f_n]_\alpha < 1$ implies that $|f_n(x) - f_n(y)| < |x-y|^\alpha$ for all $x, y \in [0,1]$, $x \neq y$. Due to the latter $(f_n)_{n\in\mathbb{N}}$ is equicontinuous and the Arzela-Ascoli theorem then yields the convergence of a subsequence in $(C([0,1]), \|\cdot\|_{C_b})$.

2. For $f \in C^\beta([0,1])$, $x, y \in [0,1]$, $x \neq y$, we have $|x-y|^{-\alpha}|f(x)-f(y)| = |x-y|^{-\beta}|x-y|^{\beta-\alpha}|f(x)-f(y)| \leq 2^{\beta-\alpha}[f]_\beta$ and hence, $\|f\|_{C^\alpha} \leq 2^{\beta-\alpha}\|f\|_{C^\beta}$. Let $0 < \alpha < \beta < 1$. If $(f_n)_{n\in\mathbb{N}} \subset B^\beta$ then as above the Arzela-Ascoli theorem then yields a convergent subsequence $f_{n_k} \to f$, $k \to \infty$, in $(C([0,1]), \|\cdot\|_{C_b})$. Moreover, since for $x, y \in [0,1]$, $x \neq y$, $|x-y|^{-\alpha}|f(x)-f(y)| \leq \sup_{k\in\mathbb{N}}[f_{n_k}]_\alpha \leq 2^{\beta-\alpha}\sup_{k\in\mathbb{N}}[f_{n_k}]_\beta \leq 2^{\beta-\alpha}$, we obtain $f \in C^\alpha([0,1])$. Finally, $[f_{n_k} - f]_\alpha = \sup_{x,y\in[0,1],x\neq y}(|x-y|^\beta|f_{n_k}(x) - f(x) - f_{n_k}(y) + f(y)|)^{\alpha/\beta}|f_{n_k}(x) - f(x) - f_{n_k}(y) + f(y)|^{1-\alpha/\beta} \leq [f_{n_k} - f]_\beta^{\alpha/\beta}(2\|f_{n_k} - f\|_{C_b})^{1-\alpha/\beta} \to 0$ as $k \to \infty$ since $[f_{n_k} - f]_\beta$ is bounded.

Exercise 2

1. Let $\phi(t) = e^{-1/t}$ for $t \in (0,\infty)$ and zero otherwise. Then $\phi \in C^\infty(\mathbb{R})$ since $\phi^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. If we set $\alpha := \int_{\mathbb{R}^d}\phi(1-|x|^2)dm^d$ and $\eta = \alpha^{-1}\phi(1-|x|^2)$ then η satisfies the conditions.

2. If $v \in \mathbb{R}^d$, $|v| \leq 1$, $\delta \in (0,1)$, $x \in \mathbb{R}^d$, $R > 0$ such that $|x| < R$, then $\delta^{-1}(f_r(x+\delta v) - f_r(x)) = \int_{B_{1+r+R}(0)}\delta^{-1}(\eta_r(x+\delta v-y) - \eta_r(x-y))f(y)dy$ and from $|\delta^{-1}(\eta_r(x+\delta v-y) - \eta_r(x-y))| \leq \|\nabla\eta_r\|_{L^\infty(\mathbb{R}^d)}$ and $f \in L^1(B_{1+r+R}(0))$ we deduce with the dominated convergence theorem that $\lim_{\delta\to 0^+}\delta^{-1}(f_r(x+\delta v) - f_r(x)) = f * (\sum_{j=1}^d v_j(\partial_{x_j}\eta_r))$. Inductively we obtain $D^\alpha(\eta_r * f) = (D^\alpha\eta_r) * f$ for all $\alpha \in \mathbb{N}_0^d$, which shows $f_r \in C^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$. Moreover, $\|f_r - f\|_{L^p} = (\int_{\mathbb{R}^d}|\int_{B_r(0)}\eta_r(h)(f(x+h) - f(x))dh|^p dx)^{1/p} \leq (\int_{\mathbb{R}^d}\|\eta_r^{1/p'}\|_{L^{p'}}^p\|\eta_r^{1/p}(f(\cdot + h) - f)\|_{L^p}^p dx)^{1/p} \leq \|\eta_r\|_{L^1}^{1/p'}(\int_{B_r(0)}(\int_{\mathbb{R}^d}|f(x+h) - f(x)|^p dx)dh)^{1/p} \leq \sup_{h\in B_r(0)}\|f(\cdot+h) - f\|_{L^p}(\int_{B_r(0)}\eta_r dh)^{1/p}$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, the claim follows from Lemma 3.38.

Exercise 3

1. Let $\epsilon = 1$ and let $\delta, R > 0$ accordingly. For all $f \in C$ and $|h| < \delta$ we have $\|f\chi_{B_R(0)}\|_{L^p} \leq \|(f(\cdot+h) - f)\chi_{B_R(0)}\|_{L^p} + \|f(\cdot+h)\chi_{B_R(0)}\|_{L^p} \leq \|f(\cdot+h) - f\|_{L^p} + \|f\chi_{B_R(h)}\|_{L^p} \leq 1 + \|f\chi_{B_R(h)}\|_{L^p}$. Inductively $\|f\chi_{B_R(0)}\|_{L^p} \leq N + \|f\chi_{B_R(Nh)}\|_{L^p}$ for all $N \in \mathbb{N}$ and by choosing N so large that $B_R(Nh) \cap B_R(0) = \emptyset$ we obtain $\|f\|_{L^p} \leq \|f\chi_{B_R(0)}\|_{L^p} + \|f\chi_{\mathbb{R}^d\setminus B_R(0)}\|_{L^p} \leq N + 2$ for all $f \in C$.

2. Let $f \in L^p(\mathbb{R}^d)$ satisfying $\operatorname{Supp} f \subset B_{1/2}(0)$, $|f| \leq 1$, $\|f\|_{L^p} \neq 0$. Moreover we write $\mathbb{Z}^d = (z_n)_{n\in\mathbb{N}}$ and define $f_n = f(\cdot - z_n)$. Then $\operatorname{Supp} f_n \subset B_{1/2}(z_n)$ are pairwise disjoint for all $n \in \mathbb{N}$ and $\|f_n\|_{L^p} = \|f\|_{L^p}$. Hence $C := \{f_n : n \in \mathbb{N}\}$ is bounded. Since $\|f_n(\cdot+h) - f_n\|_{L^p} = \|f(\cdot+h) - f\|_{L^p}$ for all $h \in \mathbb{R}^d$, condition (ii) is satisfied. Moreover, if

$\epsilon \in (0, \|f\|_{L^p})$, $R > 0$ and $n \in \mathbb{N}$ such that $B_R(0) \subset B_{1/2}(z_n) = \emptyset$ then $\|\chi_{\mathbb{R}^d \setminus B_R(0)} f_n\|_{L^p} = \|f\|_{L^p} \geq \epsilon$ and therefore, condition (iii) is not satisfied. Indeed, C is not relatively compact since the supports of its elements are disjoint.

3. Let $f \in C_c(\mathbb{R}^d)$, $f \not\equiv 0$, $\operatorname{Supp} f \subset B_1(0)$ and set $f_n(x) = n^{d/p} f(nx)$, $n \in \mathbb{N}$. Note that $\operatorname{Supp} f_n \subset B_{1/n}(0)$ and $\|f_n\|_{L^p} = \|f\|_{L^p}$. Thus, $C := \{f_n : n \in \mathbb{N}\}$ satisfies (i). Also note that (iii) is satisfied since if $\epsilon > 0$ we choose $R \geq 1$ and then for all $n \in \mathbb{N}$, $f_n \chi_{\mathbb{R}^d \setminus B_R(0)} = 0$. In order to show that (ii) is not satisfied, let $h \in \mathbb{R}^d$ such that $|h| \geq \frac{2}{n}$. Then the supports of f_n and $f_n(\cdot + h)$ are disjoint for $n \in \mathbb{N}$, since $\operatorname{Supp} f(n\cdot) \subset B_{1/n}(0)$, $\operatorname{Supp} f(n \cdot + nh) \subset B_{1/n}(-h)$ and $B_{1/n}(0) \cap B_{1/n}(-h) = \emptyset$. We need to show that $\exists \epsilon > 0\ \forall \delta > 0\ \exists h \in B_\delta(0)\ \exists n \in \mathbb{N}$ such that $\|f_n(\cdot + h) - f_n\|_{L^p} \geq \epsilon$. Let $\epsilon \in (0, 2^{1/p}\|f\|_{L^p})$. If $\delta > 0$ we choose $n \in \mathbb{N}$ such that $\frac{2}{n} < \delta$ and $h \in \mathbb{R}^d$ such that $\delta > |h| \geq \frac{2}{n}$. Since $\|f_n(\cdot + h) - f_n\|_{L^p}^p = n^d \int_{\mathbb{R}^d} |f(nx + nh) - f(nx)|^p dx = n^d (\int_{B_{1/n}(0)} |f(nx)|^p dx + \int_{B_{1/n}(-h)} |f(nx + nh)|^p dx) = 2n^d \int_{B_{1/n}(0)} |f(nx)|^p dx = 2\|f\|_{L^p}^p$, it follows that $\|f_n(\cdot + h) - f_n\|_{L^p} = 2^{1/p}\|f\|_{L^p} \geq \epsilon$.

Exercise 4

First note that since X is σ compact we may assume that X is compact. Indeed, let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets such that $K_n \subset \mathring{K}_{n+1}$ for all $n \in \mathbb{N}$ and $X = \cup_{n=1}^\infty K_n$. Let $n_0 \in \mathbb{N}$ such that $K \subset \mathring{K}_{n_0}$. We may also assume that for each $1 \leq j \leq N$ there exists some $N_j \in \mathbb{N}$ such that $U_j \subset \cup_{n=1}^{N_j} K_n$, otherwise we replace U_j by $U_j \cap \mathring{K}_{n_0}$. Then $\cup_{j=1}^N U_j \subset \cup_{n=1}^{N_0} K_n$, where $N_0 = \max\{N_1, ..., N_N\}$. If X is not compact then we replace X by $\cup_{n=1}^{N_0} K_n$, which is compact. For $j = 1, ..., N$ we define $A_j = K \cap (X \setminus \cup_{k=1, k\neq j}^N U_k)$. Note $A_j \subset U_j$, A_j is compact as a closed subset of a compact set. Moreover, for $j = 1, ..., N$ and $x \in X$ we define $\tilde{g}_j(x) = \frac{dist(x, X\setminus U_j)}{dist(x, X\setminus U_j) + dist(x, A_j)}$. Since $dist(\cdot, A)$ is a continuous function on X for any compact set $A \subset X$, we obtain $\tilde{g}_j \in C_b(X)$, $\operatorname{Supp} \tilde{g}_j \subset U_j$, $\tilde{g}_j = 1$ on A_j, $\tilde{g}_j \geq 0\ \forall j = 1, ..., N$. Let $\rho = \sum_{j=1}^N \tilde{g}_j$. Then $\rho(x) = 0$ iff $\tilde{g}_j(x) = 0\ \forall j = 1, ..., N$ iff $x \notin \cup_{j=1}^N U_j$. Then we define $g_j(x) = \rho(x)^{-1} \tilde{g}_j(x) \frac{dist(x, X\setminus \cup_{k=1}^N U_k)}{dist(x, X\setminus \cup_{k=1}^N U_k) + dist(x, K)}$. Note that $\frac{dist(x, X\setminus \cup_{k=1}^N U_k)}{dist(x, X\setminus \cup_{k=1}^N U_k) + dist(x, K)} = 1$ for all $x \in K$ and zero for all $x \in X \setminus \cup_{k=1}^N U_k$. Hence, if $x \in K$, $\sum_{j=1}^N g_j(x) = \sum_{j=1}^N \frac{\tilde{g}_j(x)}{\rho(x)} = 1$.

Remark: If $(X, d) = (\mathbb{R}^d, |\cdot|)$, then one can choose g_j such that $g_j \in C_c^\infty(\mathbb{R}^d)$ for $j = 1, ..., N$. Indeed, if ϕ denotes the function from Exercise 2.1, then $\tilde{\phi}(t) = \phi(t)\phi(1-t)$ satisfies $\operatorname{Supp} \tilde{\phi} = [0, 1]$, and $h(s) = (\int_{\mathbb{R}} \tilde{\phi}(t) dt)^{-1} \int_s^\infty \tilde{\phi}(t) dt$ satisfies $h(s) = 0$ for $s \in [1, \infty)$, $h(s) = 1$ for $s \in (-\infty, 0]$, $h \in C^\infty(\mathbb{R})$. For any $x \in K$ there exists $j_x \in \{1, ..., N\}$ such that $x \in U_{j_x}$ and then let $g_{x,j_x} : \mathbb{R}^d \to [0, \infty)$ be such that $\operatorname{Supp} g_{x,j_x} \subset U_{j_x}$, $g_{x,j_x}(x) \neq 0$. Such a function exists since if $\epsilon_{x,j_x} > 0$ such that $B_{\epsilon_{x,j_x}}(x) \subset U_{j_x}$ then we set $g_{x,j_x}(x') = h(\epsilon_{x,j_x}^{-2} |x' - x|^2)$, so that $g_{x,j_x} \in C_c^\infty(\mathbb{R}^d)$ and $B_{\epsilon_{x,j_x}}(x) = \{x' \in \mathbb{R}^d : g_{x,j_x}(x') \neq 0\}$. Since $(B_{\epsilon_{x,j_x}})_{x\in K}$ is an open cover of K, there exist $x_1, ..., x_m \in K$ such that $K \subset \cup_{i=1}^m B_{\epsilon_{x_i,j_i}}(x_i)$, where $j_i \in \{1, ..., N\}$ such that $x_i \in U_{j_i}$. Note that $\sum_{i=1}^m g_{x_i,j_i} \neq 0$ on K and $\rho(x) := (\sum_{i=1}^m g_{x_i,j_i}(x))^{-1} \sum_{j=1}^N \sum_{i\in\{1,...,m\}:\operatorname{Supp} g_{x_i,j_i} \subset U_j} g_{x_i,j_i}(x)$, $x \in \mathbb{R}^d$, is well-defined on $\mathbb{R}^d$ and satisfies $\rho > 0$ on K (note: We set $\sum_{i\in\{1,...,m\}:\operatorname{Supp} g_{x_i,j_i} \subset U_j} g_{x_i,j_i}(x) := 0$ if $\operatorname{Supp} g_{x_i,j_i} \not\subset U_j$ for all $i = 1, .., m$ for some $j \in \{1, ..., N\}$). Then $g_j = \rho^{-1} (\sum_{i=1}^m g_{x_i,j_i})^{-1} \sum_{i\in\{1,...,m\}:\operatorname{Supp} g_{x_i,j_i} \subset U_j} g_{x_i,j_i}$ have the desired properties.