Winter term 2022/2023 **Karlsruhe Institute of Technology**
Functional Analysis JProf. Dr. Xian Liao
Exercise sheet 1
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Exercise 1

1. $(\mathbb{K}^n, \|\cdot\|_2)$,

$$\|x\|_2 = \sqrt{\sum_{j=1}^n |x_j|^2} \qquad \forall x \in \mathbb{K}^n\,.$$

Normed vector space:

(i) $\forall x \in \mathbb{K}^n$, we have

$$\|x\|_2 = 0 \iff \sum_{j=1}^n |x_j|^2 = 0 \iff x_j = 0 \quad \forall j \in \{1, ..., n\}.$$

(ii) $\forall x, y \in \mathbb{K}^n$

$$\|x+y\|_2 = \sqrt{\sum_{j=1}^n |x_j|^2 + |y_j|^2 + 2\operatorname{Re}(x_j\overline{y_j})} \le \sqrt{\|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2} \le \|x\|_2 + \|y\|_2.$$

(iii) $\forall \lambda \in \mathbb{K}$, $x \in \mathbb{K}^n$,

$$\|\lambda x\|_2 = \sqrt{\sum_{j=1}^n |\lambda x_j|^2} \le |\lambda|\|x\|_2.$$

Completeness: Let $(x^k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{K}^n$. Let $1 \le j \le n$ and $k, m \in \mathbb{N}$. We have

$$|x_j^k - x_j^m| \le \sqrt{|x_j^k - x_j^m|^2} \le \|x^k - x^m\|_2.$$

Hence $(x_j^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$. Therefore $\exists x_j \in \mathbb{K} : x_j^k \xrightarrow{k\to\infty} x_j$. Set $x = (x_j)_{1\le j\le n} \in \mathbb{K}^n$. Then

$$|x^k - x| = \sqrt{\sum_{j=1}^n \underbrace{|x_j^k - x_j|^2}_{\longrightarrow 0}} \xrightarrow{k\to\infty} 0.$$

Hence x is the limit of $(x^k)_{k\in\mathbb{N}}$.

2.

$$\mathbb{B}(X) = \{f : X \longrightarrow \mathbb{K} : \sup_{x\in X} |f(x)| < \infty\}$$

$$\|f\|_{\mathbb{B}(X)} = \sup_{x\in X} |f(x)|$$

Normed vector space:

(i) $\sup_{x\in X} |f(x)| = 0 \iff |f(x)| = 0 \forall x \in X \iff f(x) = 0.$

(ii) $\sup_{x\in X} |f(x) + g(y)| \le \sup_{x\in X} |f(x)| + |g(x)| \le \sup_{x\in X} |f(x)| + \sup_{x\in X} |g(x)|.$

(iii) $\sup_{x\in X} |\lambda f(x)| = |\lambda| \sup_{x\in X} |f(x)|$.

Completeness: Let $(f^k)_{k\in\mathbb{K}}$ be a Cauchy sequence in $\mathbb{B}^k$. Let $x_0 \in X$ and $k, m \in \mathbb{N}$ we have

$$|f^k(x_0) - f^m(x_0)| \leq \sup_{x\in X} |f^k(x) - f^m(x)| = \|f^k - f^m\|_{\mathbb{B}(X)}.$$

Hence $(f^k(x_0))_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{K}, |\cdot|)$. Therefore it has a limit. We define

$$f(x_0) = \lim_{k\to\infty} f^k(x_0).$$

It remains to show that

$$\|f - f^k\|_{\mathbb{B}(X)} = \sup_{x\in X} |f(x) - f^k(x)| \xrightarrow{k\to\infty} 0.$$

Let $\varepsilon > 0$. Choose $N_2 \in \mathbb{N}$ so that for all $m, l > N_2$ we have

$$\sup_{y\in X} |f^m(y) - f^l(y)| < \frac{\varepsilon}{2}.$$

In particular

$$\sup_{y\in X} \sup_{m,l>N_2} |f^m(y) - f^l(y)| < \frac{\varepsilon}{2}.$$

Now let $x \in X$, $k \in \mathbb{N}$. Choose $N(x) \in \mathbb{N}$ so that for all $m > N_1$ we have

$$|f^m(x) - f(x)| < \frac{\varepsilon}{2}.$$

In particular

$$\sup_{x\in X} |f^{N(x)}(x) - f(x)| < \frac{\varepsilon}{2}.$$

If now $k > N_2$ then we have

$$\sup_{x\in x} |f^k(x) - f(x)| \leq \sup_{x\in X} |f^k(x) - f^{N(x)}(x)| + \sup_{x\in X} |f^{N(x)}(x) - f(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

3.

$$C_b(X) = \{f \in \mathbb{B}(X) : f \text{ continuous }\}.$$

Normed vector space: This is the same as (2).

Completeness: Let $(f^k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $C_b(X)$. By (2) we know that $\exists f \in \mathbb{B}(X)$ with $\|f^k - f\|_{\mathbb{B}(X)} \to 0$. We have to show that f is continuous. Let $x \in X$, $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $\|f^k - f\|_{\mathbb{B}(X)} < \frac{\varepsilon}{3}$ for all $k > N$. Suppose $x_n \xrightarrow{x\to\infty} x$. Continuity of f^N: $\exists M \in \mathbb{N}$ so that $n > M$ implies

$$|f^N(x) - f^N(x_n)| < \frac{\varepsilon}{3}.$$

Then

$$\begin{aligned}
|f(x) - f(x_n)| &\leq |f(x) - f^N(x)| + |f^N(x) - f^N(x_n)| + |f^N(x_n) - f(x_n)| \\
&\leq 2 \sup_{x\in X} |f^N(x) - f(x)| + |f^N(x) - f^N(x_n)| \\
&\leq \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.
\end{aligned}$$

and so f is continuous.

4.
$$C_0(\mathbb{R}^n) = \{f \in C_b(\mathbb{R}^n) : \lim_{|x|\to\infty} |f(x)| = 0\}.$$

Normed vector space: This is the same as (2).

Completeness: Let $(f^k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $C_0(\mathbb{R}^n)$. By (4): $\exists f \in C_b(\mathbb{R}^n)$: $\|f^k - f\|_{\mathbb{B}(\mathbb{R}^n)} \to 0$. We have to show that $\lim_{|x|\to\infty} |f(x)| = 0$. Let $\varepsilon > 0$, $|x_n| \to \infty$. Choose $N \in \mathbb{N}$ so that $\|f^N - f\|_{\mathbb{B}(X)} < \frac{\varepsilon}{2}$. Then choose $M \in \mathbb{N}$ so that $|f^N(x_n)| < \frac{\varepsilon}{2}$ for all $n \geq M$. Then for all $n \geq M$ we have

$$|f(x_n)| \leq |f^N(x_n)| + |f(x_n) - f^N(x_n)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

5.
$$c_0 = \{(x_j)_{j\in\mathbb{N}} : \lim_{j\to\infty} |x_j| = 0\}$$
$$\|(x_j)_{j\in\mathbb{N}}\|_{\ell^\infty} = \sup_{j\in\mathbb{N}} |x_j|.$$

In lecture: $\ell^\infty(\mathbb{N})$ is a Banach space. Alternatively: Take (2) with $X = \mathbb{N}$.

Normed vector space: See (2) with $X = \mathbb{N}$.

Completeness: Let $(x^k)_{k\in\mathbb{N}} = ((x^k_j)_{j\in\mathbb{N}})_{k\in\mathbb{N}}$ be a Cauchy sequence. By (2) there exists $x = (x_j)_{j\in\mathbb{N}} \in \ell^\infty(\mathbb{N}) = \mathbb{B}(\mathbb{N})$ so that $\|x^k - x\|_{\mathbb{B}(\mathbb{N})} \longrightarrow 0$. Same as (4): Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $\sup_{j\in\mathbb{N}} |x^N_j - x_j| < \frac{\varepsilon}{2}$. Find $J \in \mathbb{N}$ so that for all $j \in J$ we have $|x^N_j| < \frac{\varepsilon}{2}$. Then for all $j \geq J$,
$$|x_j| \leq |x^N_j - x_j| + |x^N_j| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

6.
$$C^k_b(U) = \{f \in C^k(U) : \|\partial^\alpha f\|_{\mathbb{B}(U)} < \infty \forall \alpha \in \mathbb{N}^n_0, |\alpha| \leq k\}.$$
$$\|f\|_{C^k_b(U)} = \max_{|\alpha|\leq k} \|\partial^\alpha f\|_{\mathbb{B}(U)},$$

$|\alpha| = \alpha_1 + ... + \alpha_n$, $\partial^\alpha = \partial^{\alpha_1}_{x_1} ... \partial^{\alpha_n}_{x_n}$.

Normed vector space:

(i) $\max_{|\alpha|\leq k} \sup_{x\in U} |\partial^\alpha f(x)| = 0 \iff |f(x)| = 0 \forall x \in X$.

(ii) $\max_{|\alpha|\leq k} \sup_{x\in U} |\partial^\alpha (f + g)(x)| \leq \max_{|\alpha|\leq k} \sup_{x\in U} |\partial^\alpha f(x)| + |\partial^\alpha g(x)|$.

(iii) $\max_{|\alpha|\leq k} \sup_{x\in U} |\partial^\alpha (\lambda f)(x) = \max_{|\alpha|\leq k} \sup_{x\in U} |\lambda||\partial^\alpha f(x)|$.

Completeness: Let $(f^m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $C^k_b(U)$. Then for every $|\alpha| \leq k$ we have $(\partial^\alpha f^m)_{m\in\mathbb{N}}$ Cauchy sequence in $C_b(U)$ and hence there exists a limit $f_\alpha \in C_b(U)$. $f := f_0$. To show: $f \in C^k(U)$ with $\partial^\alpha f = f_\alpha$. Suffices to assume $\alpha = (1, 0, ..., 0) = e_1$, otherwise iterate. Let $x \in U$, $h \in \mathbb{R}^n$. We have

$$\begin{aligned}\int_0^h f_{e_1}(x_1 + t, x_2, ..., x_n)\, dt \longleftarrow &\int_0^h \partial^{e_1} f^m(x_1 + t, x_2, ..., x_n)\, dt \\ &= f^m(x_1 + h, x_2, ..., x_n) - f^m(x_1, ..., x_n) \\ &\longrightarrow f(x_1 + h, x_2, ..., x_n) - f(x_1, ..., x_n).\end{aligned}$$

Somit gilt $h \mapsto f(x_1+h, x_2, x_n) - f(x_1, ..., x_n) \in C^1(\mathbb{R}, \mathbb{K})$ und $\partial_h f(x_1+g, ...) = f_{e_1}(x_1+h, ...)$.

7. $U \subseteq \mathbb{C}$ open set.

$$H^\infty(U) = \{f : U \longrightarrow \mathbb{C} : \ f \text{ holomorphic }, \|f\|_{\mathbb{B}(U)} < \infty\}.$$

Normed vector space: See (2).

Completeness Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence. As in (3), there exists a limit $f \in C_b(U)$ in supremum norm.

Theorem 0.1 (Weierstrass convergence theorem)**.** *If $(f_k)_{k\in\mathbb{N}} \in H^\infty(U)$ so that for every compact $K \subset U$ we have $\sup_{x\in K} |f(x) - f^k(x)| \longrightarrow 0$, then $f \in H^\infty(U)$ and $\partial^m f^k \longrightarrow \partial^m f$ for every $m \in \mathbb{N}$.*

Exercise 2

1. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed spaces. Their sum $X \oplus Y = X \times Y$ is a vector space. Then for any $1 \le p \le \infty$,

$$\|(x,y)\|_p = \|(\|x\|_X, \|y\|_Y)\|_{\ell^p} = (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}}.$$

defines a norm with which $X \oplus Y$ becomes a Banach space. All norms $\|\cdot\|_p$ are equivalent.

Normed space:

(i)

$$\|(x,y)\|_p^p = 0 \iff \begin{cases} \|x\|_X^p + \|y\|_Y^p = 0 & , p < \infty \\ \max\{\|x\|_X, \|y\|_Y\} = 0 & , p = \infty \end{cases} \iff (x,y) = (0,0).$$

(ii)

$$\begin{aligned} \|(x_1 + x_2, y_1 + y_2)\|_p &= \left\| \begin{pmatrix} \|x_1 + x_2\|_X \\ \|y_1 + y_2\|_Y \end{pmatrix} \right\|_{\ell^p} \\ &\le \left\| \begin{pmatrix} \|x_1\|_X \\ \|y_1\|_Y \end{pmatrix} + \begin{pmatrix} \|x_2\|_X \\ \|y_2\|_Y \end{pmatrix} \right\|_{\ell^p} \\ \text{Minkowski: } &\le \left\| \begin{pmatrix} \|x_1\|_X \\ \|y_1\|_Y \end{pmatrix} \right\|_{\ell^p} + \left\| \begin{pmatrix} \|x_2\|_X \\ \|y_2\|_Y \end{pmatrix} \right\|_{\ell^p} \\ &= \|(x_1, y_1)\|_p + \|(x_2, y_2)\|_p. \end{aligned}$$

(iii)

$$\|\lambda(x,y)\|_p = \|(|\lambda|\|x\|_X, |\lambda|\|y\|_Y)\|_{\ell^p} = |\lambda|\|(x,y)\|_p.$$

Completeness: Let $(x_j, y_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $(X \oplus Y, \|\cdot\|_p)$. For all $j,k \in \mathbb{N}$ we have

$$\|x_j - x_k\|_X = \|(x_j - x_k, 0)\|_p \le \|(x_j - x_k, y_j - y_k)\|_p$$

and the same for $\|y_j - y_k\|_Y$. Therefore $(x_j)_{j\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ are Cauchy sequences, limits $x \in X$ and $y \in X$. We have

$$\|(x_j - x, y_j - y)\|_p = \|(\|x_j - x\|_X, \|y_j - y\|_Y)\|_{\ell^p} = \begin{cases} (\|x_j - x\|_X^p + \|y_j - y\|_Y^p)^{\frac{1}{p}} & , p < \infty \\ \max\{\|x_j - x\|_X, \|y_j - y\|_Y\} & , p = \infty \end{cases} \longrightarrow 0.$$

Hence $(x_j, y_j) \longrightarrow (x,y)$ in $(X \oplus Y, \|\cdot\|_p)$.

Equivalence: Have to show: For every $p, q \in [1, \infty]$ exists $c_{p,q}, C_{p,q} > 0$ so that

$$c_{p,q} \|(x, y)\|_p \leq \|(x, y)\|_q \leq C_{p,q} \|(x, y)\|_p.$$

It suffices to show this for $q = \infty$ and all $p \in [1, \infty]$. We can assume $p < \infty$. We assume WLOG that $\|(x, y)\|_\infty = \| \max\{\|x\|_X, \|y\|_Y\} = \|x\|_X$. Clearly,

$$\|(x, y)\|_p = (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}} \leq (2\|x\|_X^p)^{\frac{1}{p}} = 2^{\frac{1}{p}} \|x\|_X.$$

On the other hand

$$\|x\|_X \leq (\|x\|_X^p)^{\frac{1}{p}} \leq (\|x\|_X^p + \|y\|_Y^p)^{\frac{1}{p}}.$$

Note: $\oplus$ is associative: Trivial for the vector space structure. For the norm:

$$\|(x, (y, z))\|_{p, X \oplus (Y \oplus Z)} = (\|x\|_X^p + (\|y\|_Y^p + \|z\|_Z^p)^{\frac{p}{p}})^{\frac{1}{p}} = \|((x, y), z)\|_{p, (X \oplus Y) \oplus Z}.$$

Then $(\mathbb{R}^n, \| \cdot \|_p) = (\mathbb{R}, \| \cdot \|_p) \oplus ... \oplus (\mathbb{R}, \| \cdot \|_p)$.

2. U closed subspace of banach space X.

$$\|\tilde{x}\|_{X/U} = \inf_{y \in U} \|y - x\|_X.$$

U Vector space clear. U normed space also clear. U complete: If $(u_n)_{n \in \mathbb{N}}$ cauchy sequence in U, it has limit $x \in X$. U closed implies $x \in U$.

Now: X/U is a vector space (LA 1).

Normed space: Well-defined? Let $\tilde{x}_1 = \tilde{x}_2$ for some $x_1, x_2 \in X$ with $x_1 - x_2 \in U$. Then

$$\|\tilde{x}_1\|_{X/U} = \inf_{y \in U} \|y - x_1\|_X = \inf_{y \in U + x_1 - x_2} \|y - x_1\|_X = \inf_{y \in U} \|y - x_2\|_X.$$

Now

(i) $\|\tilde{x}\|_{X/U} = 0 \iff \exists (y_n)_{n \in \mathbb{N}} \subset U : \|x - y_n\|_X \longrightarrow 0 \iff x \in U \iff \tilde{x} = 0.$

(ii)

$$\begin{aligned}\|\tilde{x_1} + \tilde{x_2}\|_{X/U} &= \inf_{y_1, y_2 \in U} \|x_1 + x_2 - y_1 - y_2\|_X \\ &\leq \inf_{y_1 \in U} \inf_{y_2 \in U} (\|x_1 - y_1\|_X + \|x_2 - y_2\|_X) \\ &= \|\tilde{x_1}\|_{X/U} + \|\tilde{x_2}\|_{X/U}.\end{aligned}$$

(iii) $\|\lambda \tilde{x}\|_{X/U} = \inf_{y \in U} \|\lambda x - \lambda y\|_X \leq |\lambda| \|\tilde{x}\|_{X/U}$.

Complete:

Lemma 0.2. *Let X be a normed space.*

$$X \text{ complete} \iff \left(\sum_{j \in \mathbb{N}} \|x_j\|_X < \infty \Longrightarrow \sum_{j \in \mathbb{N}} x_j \text{ converges} \right).$$

Proof. "$\Longrightarrow$". Suppose X is complete and $\sum_{j\in\mathbb{N}}|x_j|<\infty$. As for all $m\geq n\in\mathbb{N}$ we have

$$\left\|\sum_{j=1}^{n}x_j-\sum_{j=1}^{m}x_j\right\|_X\leq\left\|\sum_{j=n+1}^{m}x_j\right\|_X\leq\sum_{j=n+1}^{m}\|x_j\|_X$$

we see that $(\sum_{j=1}^{n}x_j)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence convergent.
"$\Longleftarrow$". Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Take a subsequence x_{n_k} so that $\|x_{n_{k+1}}-x_{n_k}\|<2^{-k}$ for all k. We have

$$\sum_{k\in\mathbb{N}}\|x_{n_{k+1}}-x_{n_k}\|<\infty$$

and so there exists a limit $y\in X$ so that

$$y=\lim_{m\to\infty}\sum_{k=1}^{m}(x_{n_{k+1}}-x_{n_k})=\lim_{m\to\infty}(x_{n_{m+1}}-x_{n_1}).$$

So $(x_{n_k})_{k\in\mathbb{N}}$ has limit $y+x_{n_1}$. But then we have

$$\|x_m-y-x_{n_1}\|_X\leq\|x_m-x_{n_k}\|_X+\|x_{n_k}-y-x_{n_1}\|_X,$$

and for any $\varepsilon>0$ we can find $N\in\mathbb{N}$ so that for all $m,k\geq N$ the right hand side is less than ε ($(x_k)_{k\in\mathbb{N}}$ Cauchy and x_{n_k} converges to $y-x_{n_1}$). $\square$

Let $\sum_{n\in\mathbb{N}}\|\tilde{x_n}\|_{X/U}<\infty$ be an absolutely convergent series in X/U. We can choose $u_n\in U$ so that

$$\|x_n-u_n\|_X\leq 2\|\tilde{x_n}\|_{X/U}.$$

Now set $z_n=x_n-u_n$ and note that

$$\sum_{j\in\mathbb{N}}\|z_n\|_X<\infty.$$

Therefore by the Lemma and completeness of X, we get that $(\sum_{j=1}^{n}z_j)_{n\in\mathbb{N}}$ has a limit $z\in X$. Then

$$\begin{aligned}\left\|\tilde{z}-\sum_{j=1}^{n}\tilde{x_j}\right\|_{X/U}&=\inf_{u\in U}\left\|\tilde{z}-\sum_{j=1}^{n}z_j-\sum_{j=1}^{n}(x_j-z_j)-u\right\|_X\\&=\inf_{u\in U}\left\|z-\sum_{j=1}^{n}z_j-\underbrace{\sum_{j=1}^{n}u_n-u}_{\in U}\right\|_X\\&\leq\left\|z-\sum_{j=1}^{n}z_j-\underbrace{0}_{\in U}\right\|_X\xrightarrow{n\to\infty}0.\end{aligned}$$