Exercise 1

$$L(X,Y) = \{T : X \longrightarrow Y \text{ linear and continuous }\}.$$

LA1: $(\{T : X \longrightarrow Y \text{ linear }\}, +, \cdot)$ is a vector space. Let $T_1, T_2 \in L(X,Y)$, $x \in X$ and $\lambda, \mu \in \mathbb{K}$. We have to show: $T_1 + T_2, \lambda T$ and 0 are still in $L(X,Y)$, i.e. they are continuous at every $x \in X$. **Zero**. As 0 is a constant function it is trivially continuous. **Addition and Multiplication**. Let $\varepsilon > 0$, by continuity of T_1 and T_2 there exists $\delta > 0$ with $\|x-y\|_X < \delta \Longrightarrow \|T_i(x) - T_i(y)\|_Y < \frac{1}{2}\varepsilon$ for $i = 1, 2$. Then for all $x \in X$ with $\|x - y\|_X < \delta$ we have

$$\|(T_1+T_2)(x)-(T_1-T_2)(y)\|_Y = \|T_1(x)-T_1(y)+T_2(x)-T_2(y)\| \leq \|T_1(x)-T_1(y)\|_Y+\|T_2(x)-T_2(y)\|_Y < \varepsilon.$$

If we choose $\delta > 0$ small enough so that instead $\|T_1(x) - T_1(y)\|_Y < \frac{1}{|\lambda|}\varepsilon$ for all $y \in Y$ with $\|x - y\|_X < \delta$, then

$$\|\lambda T_1(x) - \lambda T_1(y)\|_Y = |\lambda|\|T_1(x) - T_2(x)\|_Y < \varepsilon.$$

Hence $T_1 + T_2$ and λT_1 are continuous maps from X to Y.

Exercise 2

1. **Addition**. Recall: The following norms on $(X \times X)$ are equivalent:

$$\|(x,y)\|_p = \begin{cases} (\|x\|_X^p + \|x\|_X^p)^{\frac{1}{p}} & , 1 \leq p < \infty \\ \max\{\|x\|, \|y\|\} & , p = \infty. \end{cases}$$

We choose the case $p = 1$ for our norm.

Let $(x_1, x_2) \in X \times X$ and $\varepsilon > 0$. Set $\delta = \varepsilon$. Let $(y_1, y_2) \in X \times X$ with $\|(x_1, x_2) - (y_1, y_2)\|_1 = \|x_1 - y_1\| + \|x_2 - y_2\| < \delta$. Then

$$\|(x_1 + x_2) - (y_1 + y_2)\| = \|(x_1 - y_1) + (x_2 - y_2)\| \leq \|x_1 - y_1\| + \|x_2 - y_2\| < \delta = \varepsilon,$$

hence $+$ is continuous at (x_1, x_2).

Multiplication. Here we choose as norm $p = \infty$, i.e. $\|(\lambda, x)\|_\infty = \max\{|\lambda|, \|x\|\}$.

Let $(\lambda, x) \in \mathbb{K} \times X$ and $\varepsilon > 0$. Set $\delta = \min\left\{1, \frac{\varepsilon}{(1+\|x\|+|\lambda|)}\right\}$. Let $(\mu, y) \in \mathbb{K} \times X$ with $\|(\lambda, x) - (\mu, y)\|_\infty = \max\{|\lambda - \mu|, \|x - y\|\} < \delta$. Then

$$\|\lambda x - \mu y\| = \|\lambda(x - y) + (\lambda - y)y\| \leq |\lambda|\|x - y\|_Y + |\lambda - \mu|\|y\|$$
$$\leq |\lambda|\delta + \delta(\|x - y\| + \|x\|) \leq \delta(|\lambda| + \delta + \|x\|) < \varepsilon.$$

hence $\cdot$ is continuous at (λ, x).

Norm. Let $x \in X$ and $\varepsilon > 0$. Choose $\delta < \varepsilon$ and let $y \in X$ with $\|x - y\| < \delta$. By the reverse triangle inequality

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| < \delta = \varepsilon$$

we see that $\|\cdot\|$ is continuous at x.

Reverse triangle inequality: Assume without loss of generality $\|x\| \leq \|y\|$. Then

$$\big|\|x\| - \|y\|\big| = \|x - y + y\| - \|y\| \leq \|x - y\| + \|y\| - \|y\| = \|x - y\|.$$

2. We choose for the norm $p = \infty$, i.e. $\|(x_1, x_2)\|_\infty = \max\{\|x_1\|, \|x_2\|\}$. Let $\varepsilon > 0$ and $(x_1, x_2) \in X \times X$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{1+2\|(x_1,x_2)\|}\right\}$ and let $(y_1, y_2) \in X \times X$ so that $\|(x_1, x_2) - (y_1, y_2)\|_2 < \delta$. We have

$$|\langle x_1, x_2\rangle - \langle y_1, y_2\rangle| = |\langle x_1, x_2 - y_2\rangle + \langle x_1 - y_1, y_2\rangle| \leq |\langle x_1, x_2 - y_2\rangle| + |\langle x_1 - y_1, y_2\rangle|$$

$$\leq \|x_1\|\|x_2 - y_2\| + \|x_1 - y_1\|\|y_2\| \leq \delta(\|x_1\| + \|x_2 - y_2\| + \|x_2\|) \leq \delta(2\|(x_1, x_2)\|_\infty + \delta) < \varepsilon.$$

Hence $\langle \cdot, \cdot, \rangle$ is continuous in (x_1, x_2).

Exercise 3

Lemma 0.1. *Let X, Y be normed spaces and $T \in L(X, Y)$. Then the following are equivalent:*

(i) T is continuous.

(ii) T is bounded, that is there exists $K > 0$ so that $\|T(x)\|_Y \leq K\|x\|_X$ for all $x \in X$. We define

$$\|T\|_{X\to Y} = \sup_{x\in X\setminus\{0\}} \frac{\|T(x)\|_Y}{\|x\|_X}.$$

1. Linearity is easily verified. For T_R we have

$$\|T_R x\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |(T_R x)_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=2}^{\infty} |x_{n-1}|^p\right)^{\frac{1}{p}} = \|x\|_{\ell^p}.$$

Therefore T_R is bounded and hence continuous, with operator norm $\|T_R\|_{X\to Y} = 1$.

For T_L we have

$$\|T_L x\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |(T_L x)_n|^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |x_{n+1}|^p\right)^{\frac{1}{p}} = \left(\sum_{n=2}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \leq \|x\|_{\ell^p}.$$

Therefore T_L is bounded and hence continuous with operator norm $\|T_L\|_{X\to Y} \leq 1$. If we set x is a sequence with $x_1 = 0$ then the inequality above becomes an equality, hence also $\|T_L\|_{X\to Y} \geq 1$ and so $\|T_L\|_{X\to Y} = 1$.

2. $X = \{f \in C([0, 1]) : f(1) = 0\}$ with supremum norm.

$$Tf = \int_0^1 f(s)\, ds.$$

T is well-defined as continuous functions are Riemann-integrable on compact intervals. T is linear follows form the linearity of the integral. We have

$$|Tf| = \left|\int_0^1 f(s)\, ds\right| \leq \int_0^1 |f(s)|\, ds \leq 1 \cdot \max_{\{s\in[0,1]\}} |f(s)| = \|f\|_{\mathbb{B}([0,1])}$$

Hence T is bounded with operator norm ≤ 1. Define

$$f_n(x) = \begin{cases} 1 & , 0 \leq x < 1 - \frac{1}{n} \\ 1 - n(1 - \frac{1}{n} - x) & , \frac{1}{n} \leq x \leq 1 \end{cases}.$$

Then $f_n \in C([0,1])$ and $f(1) = 0$, so $f_n \in X$ with $\|f_n\|_{\mathbb{B}([0,1])} = 1$. We compute

$$\left| \int_0^s f_n(s)\, ds \right| = 1 \cdot (1 - \frac{1}{n}) + \frac{1}{2} \cdot 1 \cdot \frac{1}{n} = 1 - \frac{1}{2n} \xrightarrow{n \to \infty} 1.$$

Hence $\|T\|_{X \to Y} \geq 1$ and so $\|T\|_{X \to Y} = 1$.

Remark: Here we have that the sup is not a max in $\|T\|_{X \to Y}$.

3. $k \in C([0,1]^2; \mathbb{K})$, $X = Y = C([0,1])$ with supremum norm.

$$(Tf)(t) = \int_0^1 k(t,s) f(s)\, ds.$$

Well-defined? Continuous functions are Riemann Integrable on compact intervals, so $Tf(t)$ exists. we have to show $Tf \in C([0,1])$. Let $\varepsilon > 0$. Let $\varepsilon > 0$. As k is continuous on a compact interval it is uniformly continuous and there exists $\delta > 0$ so that

$$|(t,s) - (t',s')| < \delta \Longrightarrow |k(t,s) - k(t',s')| < \frac{\varepsilon}{\|f\|\mathbb{B}([0,1])}.$$

If now $|t - t'| < \delta$ then

$$|Tf(t) - Tf(t')| \leq \int_0^1 |k(t,s) - k(t',s')||f(s)|\, ds \leq 1 \cdot \frac{\varepsilon}{\|f\|\mathbb{B}([0,1])} \|f\|_{\mathbb{B}([0,1])} = \varepsilon.$$

We estimate

$$\begin{aligned} \|Tf\|_{\mathbb{B}(X)} &= \sup_{t \in [0,1]} \left| \int_0^1 k(t,s) f(s)\, ds \right| \\ &\leq \sup_{t \in [0,1]} \int_0^1 |k(t,s)|\, ds \|f\|_{\mathbb{B}([0,1])} \end{aligned}$$

Hence T is bounded with norm $\|T\|_{X \to Y} \leq \sup_{t \in [0,1]} \int_0^1 |k(t,s)|\, ds := A$. We claim $\geq$ also holds. By continuity of $t \mapsto \int_0^t |k(t,s)|\, ds$ and compactness of $[0,1]$ there exists $t_0 \in [0,1]$ with $A = \int_0^1 |k(t_0,s)|\, ds$. We would like to define $f = \operatorname{sign}(k(t_0,s))$ but this is not continuous. Define

$$f_n(s) = \frac{k(t_0,s)}{|k(t_0,s) + \frac{1}{n}|}$$

Then

$$\|Tf\|_{\mathbb{B}([0,1])} \geq |Tf_n(t_0)| = \int_0^1 \frac{k(t_0,s)^2}{|k(t_0,s)| + \frac{1}{n}}\, ds = \int_0^1 |k(t_0,s)| \left(1 - \frac{\frac{1}{n}}{|k(t_0,s)| + \frac{1}{n}} \right) ds$$

$$= \int_0^1 |k(t_0,s)|\, ds - \frac{1}{n} \int_0^1 \frac{|k(t_0,s)|}{|k(t_0,s)| + \frac{1}{n}}\, ds \xrightarrow{n \to \infty} A.$$

Hence $\|T\|_{X \to Y} \geq A$.

Exercise 4

(i) The function $\langle \cdot, \cdot \rangle$ is well-defined because continuous functions are integrable on compact intervals. We have

$$\langle \lambda f + \mu g, h \rangle = \int_{-1}^{1} (\lambda f(t) + \mu g(t)) h(t) \, dt = \lambda \int_{-1}^{1} f(t) h(t) \, dt + \mu \int_{-1}^{1} h(t) f(t) \, dt = \lambda \langle f, h \rangle + \mu \langle, g, h \rangle$$

and $\langle f, f \rangle = 0 \iff f = 0$ as well as $\langle f, g = \langle g, f \rangle$. Therefore $\langle \cdot, \rangle$ is an inner product. It is furthermore positive definite as

$$\langle f, g \rangle = \int_{-1}^{1} f(s)^2 \, ds,$$

which is ≥ 0 in general and $= 0$ if and only if $f = 0$. Therefore we have a pre-Hilbert space. The induced norm $\|f\| = \sqrt{\int_{-1}^{1} |f(s)|^2 \, ds}$ is not complete: Set

$$f_n(t) = \begin{cases} 0 & , t \in [-1, 0] \\ nt & , t \in (0, \frac{1}{n}) \\ 1 & , t \in [\frac{1}{n}, 1] \end{cases} .$$

We show that f_n has no limit in $(C([-1,1], \mathbb{R}), \| \cdot \|)$. Suppose g is such a limit. Then

$$0 \longleftarrow \int_{-1}^{1} |g(t) - f_n(t)|^2 \, dt = \int_{-1}^{0} |g(t)|^2 \, dt + \int_{0}^{\frac{1}{n}} |g(t) - nt|^2 \, dt + \int_{\frac{1}{n}}^{1} |g(t) - 1|^2 \, dt.$$

In particular as g is continuous we have $g(x) = 0$ for $x \in [-1, 0]$ and $g(t) = 1$ for $x \in (0, 1]$. But this contradicts the continuity of g. The sequence $(f_n)_{n \in \mathbb{N}}$ is, however, a Cauchy sequence: Let $n \leq m$. Then

$$\begin{aligned}
\int_{-1}^{1} |f_n(t) - f_m(t)|^2 \, dt &= \int_{0}^{\frac{1}{m}} |nt - mt|^2 \, dt + \int_{\frac{1}{m}}^{\frac{1}{n}} |nt - 1|^2 \, dt \\
&= \int_{0}^{\frac{1}{m}} n^2 t^2 + m^2 t^2 - 2nmt^2 \, dt + \int_{\frac{1}{m}}^{\frac{1}{n}} n^2 t^2 + 1 - 2nt \, dt \\
&= n^2 \frac{1}{3} \frac{1}{m^3} + m^2 \frac{1}{3} \frac{1}{m^3} - 2nm \frac{1}{3} \frac{1}{m^3} \\
&+ n^2 \frac{1}{3} \frac{1}{n^3} + \frac{1}{n} - 2n \frac{1}{2} \frac{1}{n^2} - n^2 \frac{1}{3} \frac{1}{m^3} - \frac{1}{m} + 2n \frac{1}{2} \frac{1}{m^2} \\
&\leq \frac{1}{3} \Big(\frac{1}{m} + \frac{1}{m} + \frac{1}{n} + 3 \frac{1}{n} + 3 \frac{1}{m} \Big) \xrightarrow{n,m \to \infty} 0.
\end{aligned}$$