Winter term 2022/2023 **Karlsruhe Institute of Technology**
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Exercise 1

1. We have $0 \in A^\perp$ and if $x, y \in A^\perp$, $\lambda \in \mathbb{K}$, then $\langle \lambda x + y, z\rangle = \lambda\langle x, z\rangle + \langle y, z\rangle = 0$ for all $z \in A$. If $(x_n)_{n\in\mathbb{N}} \subset A^\perp$ is a sequence converging to some $x \in H$, then $\langle x_n, y\rangle = 0$ for all $y \in A$, $n \in \mathbb{N}$. By the continuity of the inner product it follows that $\langle x, y\rangle = 0$ for all $y \in A$.

2. If $x \in B$, then $\langle x, y\rangle = 0$ for all $y \in B^\perp$, and therefore $x \in (B^\perp)^\perp$. Let $x \in (B^\perp)^\perp$ and $p : H \to B$ be the projection from the Hilbert projection theorem. Since $x - p(x) \in B^\perp$ and for all $y \in B^\perp$, $0 = \langle x, y\rangle = \langle x - p(x), y\rangle$, it follows that $x - p(x) = 0$, and hence $x = p(x) \in B$.

Exercise 2

1. Since $(x^2 - 1)^n = \sum_{j=0}^n \binom{n}{j} x^{2j}(-1)^{n-j}$, the leading term of $P_n(x)$ is $\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^{2n}) = \frac{(2n)!}{2^n(n!)^2}x^n$. In order to show that $P_n(1) = 1$, we prove that $\frac{d^n}{dx^n}((x^2-1)^n)|_{x=1} = 2^n n!$ for all $n \in \mathbb{N}_0$. We have $P_0 \equiv 1$ and $P_1(x) = x$. If the equality is true for some $n \in \mathbb{N}$, then $\frac{d^{n+1}}{dx^{n+1}}((x^2-1)^{n+1}) = 2(n+1)\frac{d^n}{dx^n}((x^2-1)^n x) = 2(n+1)(\frac{d^n}{dx^n}((x^2-1)^n)x + \frac{d^{n-1}}{dx^{n-1}}((x^2-1)^n))$. We obtain $\frac{d^{n+1}}{dx^{n+1}}((x^2-1)^{n+1})|_{x=1} = 2(n+1)\cdot 2^n n! = 2^{n+1}(n+1)!$. Let $0 \le m < n$. After m integration by parts we arrive at $\int_{-1}^1 x^m \frac{d^n}{dx^n}((x^2-1)^n)dx = (-1)^m m! \int_{-1}^1 \frac{d^{n-m}}{dx^{n-m}}((x^2-1)^n)dx = (-1)^m m! [\frac{d^{n-m-1}}{dx^{n-m-1}}((x^2-1)^n)]_{-1}^1 = 0$.

2. Using integration by parts we compute $\int_{-1}^1 (P_n(x))^2 dx = \frac{1}{2^n n!}\frac{(2n)!}{2^n(n!)^2}\int_{-1}^1 x^n \frac{d^n}{dx^n}((x^2-1)^n)dx = \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^1 (1-x^2)^n dx$, where $\int_{-1}^1 (1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx = \int_0^1 (1-s)^n s^{-1/2} ds = \frac{2^{2n}(n!)^2}{(2n)!}\int_0^1 s^{\frac{2n-1}{2}} = \frac{2}{2n+1}\frac{2^{2n}(n!)^2}{(2n)!}$.

Exercise 3

1. Let $1 \le p < \infty$. Define $A = \{x \in l^p(\mathbb{N}) : x_j \in \mathbb{Q}\ \forall j \in \mathbb{N} \text{ and } \exists J \in \mathbb{N}\ \forall j \ge J : x_j = 0\}$ and $B = \{x \in l^p(\mathbb{N}) : \exists J \in \mathbb{N}\ \forall j \ge J : x_j = 0\}$. Note that A is countable. Let $x \in l^p(\mathbb{N})$ and define $(x^k)_{k\in\mathbb{N}}$ by $x_j^k = x_j$ if $j \le k$, and $x_j^k = 0$ otherwise. Then $x^k \to x$ in $l^p(\mathbb{N})$ as $k \to \infty$. Let $\epsilon > 0$ and $k \in \mathbb{N}$ such that $\|x - x^k\|_{l^p(\mathbb{N})} < \frac{\epsilon}{2}$. Moreover, there exists a $y \in A$ such that $\|y - x^k\|_{l^p(\mathbb{N})} < \frac{\epsilon}{2}$. Hence, $\|x - y\|_{l^p(\mathbb{N})} \le \|x - x^k\|_{l^p(\mathbb{N})} + \|x^k - y\|_{l^p(\mathbb{N})} < \epsilon$. To show that $l^\infty(\mathbb{N})$ is not separable, let $A = (x^k)_{k\in\mathbb{N}} \subset l^\infty(\mathbb{N})$ be countable. For $j \in \mathbb{N}$, set $y_j = x_j^j + 1$ if $|x_j^j| \le 1$, and $y_j = 0$ otherwise. Then $y = (y_j)_{j\in\mathbb{N}} \in l^\infty(\mathbb{N})$ and $\|y - x^k\|_{l^\infty(\mathbb{N})} \ge 1$ for all $k \in \mathbb{N}$. Therefore, y can not be approximated by elements in A.

2. For $x, y \in \mathbb{R}$ we set $e_y^x = 1$ if $x = y$, and zero otherwise. $(e_y^x)_{y\in\mathbb{R}}$ is an orthonormal system in $l^2(\mathbb{R})$. The pairwise distance is $\sqrt{2}$, and therefore, the family of balls $(B_{\sqrt{2}}(e_y^x))_{y\in\mathbb{R}}$ is an uncountable collection of disjoint sets. Hence, there cannot be a dense countable subset.

Exercise 4

If $(x_j)_{j\in\mathbb{N}}$ is an orthonormal basis, then by the proof of Theorem 2.22, Parseval's identity holds. If we assume that $(x_j)_{j\in\mathbb{N}}$ is and orthonormal set for which Parseval's identity holds, we take $x \in H$ such that $\langle x, x_j\rangle = 0$ for all $j \in \mathbb{N}$. Since $\|x\| = \sum_{j=1}^{\infty} |\langle x, x_j\rangle|^2 = 0$, it follows that $x = 0$, and hence by definition, $(x_j)_{j\in\mathbb{N}}$ is an orthonormal basis.

Exercise 5

1. Let J_1 denote the Riesz isomorphism of H_1 and let $y \in H_2$. Note that the map $\varphi_y = (H_1 \ni x \mapsto \langle Tx, y\rangle)$ is contained in H_1^* so that $\tilde{y} = J_1^{-1}(\varphi_y)$ satisfies $\langle Tx, y\rangle = \langle x, \tilde{y}\rangle$ for all $x \in H_1$. Set $T^*y := \tilde{y}$. Then $T^* \in L(H_2, H_1)$ and it has the desired property. To show uniqueness, let $T' \in L(H_2, H_1)$ be another operator satisfying the equality. It follows that for all $x \in H_1$ and $y \in H_2$, $\langle x, T'y\rangle = \langle x, T^*y\rangle$ and hence, $T'y = T^*y$ for all $y \in H_2$.
2. For all $x \in H_1$ we have $\|Tx\|^2 = \langle x, T^*Tx\rangle \leq \|x\|\|T^*\|\|Tx\|$, which implies that $\|Tx\| \leq \|T\|\|x\|$, and thus $\|T\| \leq \|T^*\|$. For all $y \in H_2$ we have $\|T^*x\|^2 = \langle TT^*y, y\rangle \leq \|T\|\|T^*y\|\|y\|$, and therefore similarly $\|T^*\| \leq \|T\|$.