Winter term 2022/2023 **Karlsruhe Institute of Technology**
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Exercise 1

We define $A : H \to H$, $x \mapsto J^{-1}(\overline{Q(x,\cdot)})$, where J denotes the Riesz isomorphism. A satisfies $Q(x,y) = \langle Ax, y\rangle$ for all $x, y \in H$, and $\|Ax\|^2 = |Q(x, Ax)| \le C\|x\|\|Ax\|$, which implies that $A \in L(H)$ with $\|A\|_{H\to H} \le C$. A is injective since for all $x \in H$, $\langle Ax, x\rangle = Q(x,x) \ge \delta\|x\|^2$, and therefore $\|Ax\| \ge \delta\|x\|$. Moreover, this inequality implies that A has closed range. Indeed, if $(x_n)_{n\in\mathbb{N}} \subset H$ is such that $(Ax_n)_{n\in\mathbb{N}}$ converges to some $y \in H$, then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence by the inequality, and since H is complete, it converges to some $x \in H$. Hence, $\|y - Ax\| = \lim_{n\to\infty} \|Ax_n - Ax\| = 0$ since A is bounded. If $z \in (\mathrm{Ran}(A))^\perp$, then $0 = \langle Az, z\rangle \ge \delta\|z\|^2$, and thus, $(\mathrm{Ran}(A))^\perp = \{0\}$. Since the range is a closed subspace, it follows that $\mathrm{Ran}(A) = H$. Therefore, there exists $A^{-1} : H \to H$, and from $\|Ax\| \ge \delta\|x\|$ it follows that $\|x\| \ge \delta\|A^{-1}x\|$ for all $x \in H$, which implies that A^{-1} is bounded and $\|A^{-1}\|_{H\to H} \le \delta^{-1}$.

Exercise 2

1. For $x, y \in l^2(\mathbb{N})$ we have $\langle T_L x, y\rangle = \sum_{j=1}^\infty x_{j+1}\overline{y_j} = \langle x, T_R y\rangle$, where T_R is the right shift operator. Hence, $T_L^* = T_R$. Moreover, $\langle T_R x, y\rangle = \sum_{j=2}^\infty x_j\overline{y_{j-1}} = \langle x, T_L y\rangle$, which implies that $T_L^{**} = T_L$.

2. T is linear since the integral is linear. For $f \in L^2([0,1])$ we have by the Cauchy-Schwarz inequality, $\int_0^1 |Tf(t)|^2 dt \le \int_0^1 (\int_0^t |f(s)|ds)^2 dt \le \int_0^1 \|f\|_{L^2}^2 dt = \|f\|_{L^2}^2$, and hence $\|Tf\|_{L^2} \le \|f\|_{L^2}$. Let $f, g \in L^2([0,1])$). We have $\langle Tf, g\rangle = \int_0^1\int_0^t f(s)ds\overline{g(t)}dt = \int_0^1\int_s^1 f(s)\overline{g(t)}dtds = \langle f, T^*g\rangle$, where $T^*g(s) = \int_s^1 g(t)dt$ for $s \in [0,1]$.

3. For $f \in L^2([0,1];\mathbb{R})$ and $g \in L^2([0,1];\mathbb{C})$ we compute $\langle T_k f, g\rangle = \int_0^1\int_0^1 k(t,s)f(s)ds\overline{g(t)}dt = \int_0^1 f(s)\int_0^1 k(t,s)\overline{g(t)}dtds$. If we set $k^*(t,s) := \overline{k(s,t)}$ for almost every $s, t \in [0,1]$, then $T_k^* = T_{k^*}$.

Exercise 3

1. If $\|f\|_{L^{p,q}} = 0$, then $f^*(s) = 0$ for almost every $s > 0$, and hence, $0 = d_f(s) = m^n(\{x \in \mathbb{R}^n : |f(x)| > 0\})$, which implies $f = 0$ almost everywhere. For $\lambda \in \mathbb{C}\setminus\{0\}$ and $\alpha \ge 0$ we have $d_{\lambda f}(\alpha) = m^n(\{x \in \mathbb{R}^n : |\lambda f(x)| > \alpha\}) = d_f(|\lambda|^{-1}\alpha)$. It follows that for $s > 0$, $(\lambda f)^*(s) = \inf\{\alpha > 0 : d_f(|\lambda|^{-1}\alpha) \le s\} = |\lambda| f^*(s)$. Let $f, g \in L^{p,q}$ and $s_1, s_2 > 0$. We set $A = \{\alpha_1 > 0 : d_f(\alpha_1) \le s_1\}$, $B = \{\alpha_2 > 0 : d_g(\alpha_2) \le s_2\}$ and $S = \{\alpha > 0 : d_{f+g}(\alpha) \le s_1 + s_2\}$. Since $d_{f+g}(\alpha_1 + \alpha_2) \le d_f(\alpha_1) + d_g(\alpha_2)$ for $\alpha_1, \alpha_2 \ge 0$, we have $A + B \subset S$. Hence, $(f+g)^*(s_1+s_2) = \inf S \le \inf A + \inf B = f^*(s_1) + g^*(s_2)$. Using this we compute $\|f+g\|_{L^{p,q}} \le \|s^{\frac{1}{p}-\frac{1}{q}}(f^*(\frac{\cdot}{2}) + g^*(\frac{\cdot}{2}))\|_{L^q(0,\infty)} \le 2^{\frac{1}{p}-\frac{1}{q}}(\|(\frac{s}{2})^{\frac{1}{p}-\frac{1}{q}} f^*(\frac{\cdot}{2})\|_{L^q(0,\infty)} + \|(\frac{s}{2})^{\frac{1}{p}-\frac{1}{q}} g^*(\frac{\cdot}{2})\|_{L^q(0,\infty)}) = 2^{\frac{1}{p}}(\|f\|_{L^{p,q}} + \|g\|_{L^{p,q}})$.

2. Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. Since $\|f\|_{L^p}^p = \int_{\mathbb{R}^n}\int_0^{|f(x)|} ps^{p-1}dsdx = p\int_0^\infty s^{p-1}d_f(s)ds$, it suffices to show $d_f = d_{f^*}$. To this end we first note that $\alpha < f^*(s)$ if and only if $s < d_f(\alpha)$. Hence, $d_{f^*}(\alpha) = m^n(\{s > 0 : f^*(s) > \alpha\}) = m^n((0, d_f(\alpha))) = d_f(\alpha)$.

3. Let $q_1 < q_2 = \infty$ and $f \in L^{p,q_1}$. For $s > 0$ we have $|s^{\frac{1}{p}} f^*(s)| = (\frac{q_1}{p} \int_0^s (t^{\frac{1}{p}-\frac{1}{q_1}} |f^*(s)|)^{q_1} dt)^{\frac{1}{q_1}} \leq (\frac{q_1}{p} \int_0^s (t^{\frac{1}{p}-\frac{1}{q_1}} |f^*(t)|)^{q_1} dt)^{\frac{1}{q_1}} \leq (\frac{q_1}{p})^{\frac{1}{q_1}} \|f\|_{p,q_1}$. This implies that $\|f\|_{L^{p,\infty}} \leq (\frac{q_1}{p})^{\frac{1}{q_1}} \|f\|_{p,q_1}$. Let now $q_1 < q_2 < \infty$ and $f \in L^{p,q_1}$. Using the above steps we compute $\|f\|_{L^{p,q_2}}^{q_2} = \int_0^\infty (s^{\frac{1}{p}-\frac{1}{q_1}} |f^*(s)|)^{q_1} s^{\frac{q_2-q_1}{p}} |f^*(s)|^{q_2-q_1} ds \leq \|f\|_{L^{p,q_1}}^{q_1} \|f\|_{L^{p,\infty}}^{q_2-q_1} \leq C \|f\|_{L^{p,q_1}}^{q_2}$.

4. Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be measurable and $s_1, s_2 > 0$. We set $A = \{t_1 > 0 : d_f(t_1) \leq s_1\}$, $B = \{t_2 > 0 : d_g(t_2) \leq s_2\}$ and $P = \{t > 0 : d_{fg}(t) \leq s_1 + s_2\}$. Since $d_{fg}(t_1 t_2) \leq d_f(t_1) + d_g(t_2)$ for all $t_1, t_2 > 0$, it follows that $A \cdot B \subset P$, and hence $(fg)^*(s_1 + s_2) = \inf P \leq \inf A \cdot \inf B = f^*(s_1) g^*(s_2)$. If $q < \infty$ we have $\|fg\|_{L^{p,q}} \leq (\int_0^\infty s^{\frac{q}{p}-1} |f^*(\frac{\cdot}{2}) g^*(\frac{\cdot}{2})|^q ds)^{\frac{1}{q}} \leq 2^{\frac{1}{p}} \|s^{\frac{1}{p_0}-\frac{1}{q_0}} f^*\|_{L^{q_0}(0,\infty)} \|s^{\frac{1}{p_1}-\frac{1}{q_1}} g^*\|_{L^{q_1}(0,\infty)} = 2^{\frac{1}{p}} \|f\|_{L^{p_0,q_0}} \|g\|_{L^{p_1,q_1}}$. If $q = \infty$ we have $\|fg\|_{L^{p,\infty}} \leq \|s^{\frac{1}{p}} f^*(\frac{\cdot}{2}) g^*(\frac{\cdot}{2})\|_{L^\infty(0,\infty)} \leq 2^{\frac{1}{p}} \|f\|_{L^{p_0,\infty}} \|g\|_{L^{p_1,\infty}}$.