Winter term 2022/2023 | **Karlsruhe Institute of Technology**
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Exercise 1

1. Without loss of generality let $C_0, C_1 > 0$. For $\epsilon > 0$ the function $G_\epsilon(z) = u(z)C_0^{z-1}C_1^{-z}e^{\epsilon z^2}$ is holomorphic in $\mathring{\mathcal{C}}$ and continuous on $\mathcal{C}$. For $R > 0$ let $Q_R = \mathcal{C} \cap \{z \in \mathbb{C} : |\mathrm{Im} z| \leq R\}$. The Maximum principle implies $\sup_{z\in Q_R}|G_\epsilon(z)| = \sup_{z\in\partial Q_R}|G_\epsilon(z)|$ and we compute that for $|t| \leq R$, $|G_\epsilon(it)| \leq 1$, $|G_\epsilon(1+it)| \leq e^\epsilon$, and for $x \in [0,1]$, $|G_\epsilon(x \pm iR)| \leq (\sup_{z\in\mathcal{C}}|u(z)|C_0^{x-1}C_1^{-x})e^{\epsilon(x^2-R^2)}$. Since for any $\epsilon > 0$ there exists $R_0 > 0$ such that for $R \geq R_0$, $(\sup_{z\in\mathcal{C}}|u(z)|C_0^{x-1}C_1^{-x})e^{\epsilon(x^2-R^2)} \leq 1$ and since for any $z \in \mathcal{C}$ there exists $R' \geq R_0$ such that $z \in Q_{R'}$ with $|G_\epsilon(z)| \leq e^\epsilon$, we arrive at $|u(z)| \leq |C_0^{1-z}C_1^z e^{-\epsilon z^2+\epsilon}|$ for all $z \in \mathcal{C}$, $\epsilon > 0$. Letting $\epsilon \to 0$ yields the assertion.

2. Let $f = \sum_{k=1}^n \alpha_k \chi_{A_k}$, $g = \sum_{l=1}^m \beta_l \chi_{B_l}$, where $\alpha_k, \beta_l \in \mathbb{C}\setminus\{0\}$, $A_k, B_l \in \mathcal{A}$ such that A_k are pairwise disjoint, B_l are pairwise disjoint, $\mu(A_k), \mu(B_l) \in (0,\infty)$, and assume that $\|f\|_{L^p} = \|g\|_{L^{\frac{q}{q-1}}} = 1$. We are going to show that $|\int g(Tf)d\mu| \leq C_0^{1-\theta}C_1^\theta$ which then by duality yields $Tf \in L^q$ and $\|Tf\|_{L^q} \leq C_0^{1-\theta}C_1^\theta$. To this end we define $F : \mathcal{C} \to \mathbb{C}$, $F(z) = \int g_z(Tf_z)d\mu$, where $f_z = \sum_{k=1}^n |\alpha_k|^{a(z)}\frac{\alpha_k}{|\alpha_k|}\chi_{A_k}$, $g_z = \sum_{l=1}^m |\beta_l|^{b(z)}\frac{\beta_l}{|\beta_l|}\chi_{B_l}$ and $a(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1}z$, $b(z) = \frac{q}{q-1}\frac{q_0-1}{q_0}(1-z) + \frac{q}{q-1}\frac{q_1-1}{q_1}z$. Note $f_\theta = f$, $g_\theta = g$. F is holomorphic in $\mathring{\mathcal{C}}$ and continuous and bounded on $\mathcal{C}$. Moreover, $\sup_{t\in\mathbb{R}}|F(it)| \leq C_0$, $\sup_{t\in\mathbb{R}}|F(1+it)| \leq C_1$, where we used that $\|f_{it}\|_{L^{p_0}} = \|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p} = 1$ and $\|g_{it}\|_{L^{\frac{q_0}{q_0-1}}} = \|g_{1+it}\|_{L^{\frac{q_1}{q_1-1}}} = \|g\|_{L^{\frac{q}{q-1}}} = 1$. The Three lines inequality then implies $|F(\theta)| \leq C_0^{1-\theta}C_1^\theta$.

Exercise 2

1. (i)⇒(ii): If $(x_n)_{n\in\mathbb{N}} \subset K$ does not contain a subsequence which converges in K, then for all $x \in K$ there exists some $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains only finitely many x_n. Since K is compact and $K \subset \cup_{x\in K}B_{\epsilon_x}(x)$ is an open covering there exists a finite set $F \subset K$ such that $K \subset \cup_{x\in F}B_{\epsilon_x}(x)$ which is a contradiction.

2. (ii)⇒(iii): Suppose K is not totally bounded, i.e., there exists $\epsilon > 0$ such that for any $m \in \mathbb{N}$, $x_1', ..., x_m' \in K$, K is not contained in $\cup_{j=1}^m B_\epsilon(x_j')$. Therefore if $x_1 \in K$ we can pick $x_2 \in K \setminus B_\epsilon(x_1)$ and $x_3 \in K \setminus (B_\epsilon(x_1) \cup B_\epsilon(x_2))$ and successively obtain a sequence $(x_n)_{n\in\mathbb{N}}$ which does not have a convergent subsequence in K which is a contradiction. K is complete since any Cauchy sequence which contains a convergent subsequence already converges.

3. (iii)⇒(i): Suppose there exists an open covering $\mathcal{U}$ of K which does not contain a finite covering. Let $\epsilon_1 = 1$ and select $x_1^{(1)}, ..., x_{N_1}^{(1)}$ such that $K \subset \cup_{i=1}^{N_1}B_{\epsilon_1}(x_i^{(1)})$. Hence there exists some $x_i^{(1)}$ such that $B_{\epsilon_1}(x_i^{(1)})$ can not be covered by finitely many $U \in \mathcal{U}$. For $\epsilon_n = 2^{-(n-1)}$ we can construct s_n such that $\cap_{j=1}^n B_{\epsilon_j}(s_j)$ can not be covered by finitely many $U \in \mathcal{U}$ for all $n \in \mathbb{N}$. Then $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence there exists a limit $s_0 \in K$. Write $\mathcal{U} = (U_i)_{i\in I}$ for some index set I, and choose $i_0 \in I$ such that

$s_0 \in U_{i_0}$. Since U_{i_0} is open, $\eta := \inf\{d(s, s_0) : s \notin U_{i_0}\} > 0$. If we choose $n \in \mathbb{N}$ such that $d(s_n, s_0) < \frac{\eta}{2}$ and $2^{-(n-1)} < \frac{\eta}{2}$, then $\cap_{j=1}^n B_{\epsilon_j}(s_j) \subset B_\eta(s_0) \subset U_{i_0}$ which is a contradiction.

Exercise 3

1. Let first $\mu(X \setminus B) = 0$. We define $F = \{A \subset X : A \text{ Borel set }, \forall \epsilon > 0\ \exists C \subset A : C \text{ is closed}, \mu(A \setminus C) < \epsilon\}$. Then F contains all closed sets and if $(A_j)_{j\in\mathbb{N}} \subset F$ then $\cap_{j=1}^\infty A_j, \cup_{j=1}^\infty A_j \in F$. Indeed, if $\epsilon > 0$ and $C \subset A_1$ is closed such that $\mu(A_1 \setminus C) < \epsilon$, then also $\mu(\cup_{j=1}^\infty A_j \setminus C) < \epsilon$; moreover since $\infty > \mu(\cup_{j=1}^\infty A_j) = \sum_{j=1}^\infty \mu(A_j \setminus \cup_{k=1}^{j-1} A_k)$, there exists $J \in \mathbb{N}$ such that $\mu(\cup_{j=J}^\infty (A_j \setminus \cup_{k=1}^{j-1} A_k)) < \frac{\epsilon}{2}$, and if $C_j \subset A_j$ are closed such that $\mu(A_j \setminus C_j) < \frac{\epsilon}{2J}$ for $j = 1, ..., J-1$, then $\mu(\cup_{j=1}^\infty A_j \setminus \cup_{k=1}^{J-1} C_k) < \epsilon$. Furthermore, since open sets are countable unions of closed sets, every open set is contained in F. We define $G = \{A \subset X : A, X \setminus A \in F\}$. Then G contains all complements of its elements, countable unions of its elements and $\emptyset, X \in G$. Hence, G is a sigma algebra which contains all open sets and therefore it contains the Borel sigma algebra. In particular $B \in G$. If $\mu(X \setminus B) > 0$ we replace μ by ν, where $\nu(A) := \mu(A \cap B)$ for Borel sets A. The above steps imply that for any $\epsilon > 0$ there exists a closed set $C \subset B$ such that $\mu(B \setminus C) = \nu(B \setminus C) < \epsilon$.

2. For $j \in \mathbb{N}$ let $C_j \subset X$ be compact such that $X = \cup_{j=1}^\infty C_j$. Since X is locally compact, for all $x \in X$ there exists a neighbourhood U_x of x such that $\overline{U_x}$ is compact. Since $C_1 \subset \cup_{x\in C_1} U_x$ and C_1 is compact, there exist $x_1, ..., x_m \in C_1$ such that $C_1 \subset \cup_{i=1}^m U_{x_i}$. Then $K_1 := \cup_{i=1}^m \overline{U_{x_i}}$ is compact and $C_1 \subset \mathring{K}_1$. Recursively we define K_j in the following way: Since $K_{j-1} \cup C_j$ is compact and $K_{j-1} \cup C_j \subset \cup_{x\in K_{j-1}\cup C_j} U_x$, there exist $x_1, ..., x_m \in K_{j-1} \cup C_j$ such that $K_{j-1} \cup C_j \subset \cup_{i=1}^m U_{x_i}$, and we set $K_j := \cup_{i=1}^m \overline{U_{x_i}}$. Then $(K_j)_{j\in\mathbb{N}}$ has the desired properties.