1. HILBERT SPACES

Hilbert Space Methods Seminar, SS 2020

Ionita A., Jprof. Dr. Liao X., Dr. Cossetti L.

Outline

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  - Definition and dominated convergence theorem
  - L^2 Hilbert space

Definition and elementary properties of Hilbert spaces

Def. Hilbert space $\mathcal{H}$

H1. $\mathcal{H}$ complex linear vector space.

- Standard properties for vector addition and multiplication with a scalar
- Unique zero vector $\mathbf{0} \in \mathcal{H}$

H2. $\mathcal{H}$ is equipped with a strictly positive scalar product.

- $\langle g, f \rangle = \overline{\langle f, g \rangle}$
- $\langle f, g + \alpha h \rangle = \langle f, g \rangle + \alpha \langle f, h \rangle$
- $\langle f, f \rangle > 0$ except for $f = \mathbf{0}$

⇒ ***norm***: $||f|| \coloneqq [\langle f, f \rangle]^{1/2}$

H3. $\mathcal{H}$ is complete.

- Each Cauchy sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ has a limit in $\mathcal{H}$

$$\{f_n\}_{n \in \mathbb{N}} \in \mathcal{H}: \lim_{m,n \to \infty} ||f_n - f_m|| = 0 \quad \Rightarrow \quad \exists\, f \in \mathcal{H}: \lim_{n \to \infty} ||f - f_m|| = 0$$

H4. $\mathcal{H}$ is separable.

- $\mathcal{H}$ has a countable orthonormal basis

⇒ ***dimension***: number of elements in the basis, independent of chosen basis

Elements of $\mathcal{H}$: vectors

Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of $\mathcal{H}$, $f \in \mathcal{H}$ arbitrary.

Then there exists a sequence $\{\alpha_1, \alpha_2, \ldots\}$ of complex numbers such that:

$$f = \sum_k \alpha_k \, e_k \quad \text{and} \quad ||f||^2 = \sum_k |\alpha_k|^2$$

$$\text{with coefficients } \alpha_k = \langle e_k, f \rangle$$

- finite dimensionality: as in linear algebra, $k = \overline{1, N}$; $N < \infty$
- infinite-dimensional spaces: convergence in the sense that:

$$\left\| f - \sum_{k=1}^{N} \alpha_k e_k \right\| \to 0 \quad \text{as} \quad N \to \infty$$

Elementary properties of $\mathcal{H}$

- Polarisation identity:

$$4\langle f, g\rangle = \left||f+g|\right|^2 - \left||f-g|\right|^2 - i\left||f+ig|\right|^2 + i\left||f-ig|\right|^2$$

- Schwarz inequality:

$$|\langle f, g\rangle| \leq \left||f|\right| \cdot \left||g|\right|$$

- Triangle inequality:

$$\left||f+g|\right| \leq \left||f|\right| + \left||g|\right|$$

$$\left||f+g|\right|^2 \leq 2\left||f|\right|^2 + 2\left||g|\right|^2$$

$$\left|\left||f|\right| - \left||g|\right|\right| \leq \left||f-g|\right|$$

Convergence: weak, strong

Types of convergence

- ***Def.*** *Strong convergence:* $\underset{n\to\infty}{\mathrm{s}-\lim} f_n = f$

  A sequence of vectors $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ is said to converge strongly to a vector $f \in \mathcal{H}$ if $\left|\left|f_n - f\right|\right| \to 0 \ (n \to \infty)$

- ***Def.*** *Weak convergence:* $\underset{n\to\infty}{\mathrm{w}-\lim} f_n = f$

  A sequence of vectors $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$ is said to converge weakly to a vector $f \in \mathcal{H}$ if, for each $g \in \mathcal{H}$, the sequence of complex numbers $\{\langle f_n, g\rangle\}_{n\in\mathbb{N}} \to \langle f, g\rangle \ (n \to \infty)$

Strong convergence really is stronger!

- ***Proposition 1.1.***
  - a. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathcal{H}$. Then:

$$\underset{n\to\infty}{\mathrm{s-lim}}\, f_n = f \;\Leftrightarrow\; \underset{n\to\infty}{\mathrm{w-lim}}\, f_n = f \;\; and \;\; \lim_{n\to\infty} \|f_n\| = \|f\|.$$

  - b. If $\underset{n\to\infty}{\mathrm{s-lim}}\, f_n = f$ and $\underset{n\to\infty}{\mathrm{s-lim}}\, g_n = g$, then $\lim_{n\to\infty} \langle f_n, g_n\rangle = \langle f, g\rangle$

- Example 1.2.

  Every infinite orthonormal sequence $\{e_n\}$ converges weakly to **0**, but never converges strongly.

Vector – valued functions

Definitions

- ***Def.*** *vector-valued function:* a mapping $f : \Lambda \to \mathcal{H}$
  - Λ is an interval J: $f(s)$ for $s \in J$

- ***Def.*** $f : J \to \mathcal{H}$ *strongly continuous:*

$$\forall t \in J: \lim_{s \to t,\, s \in J} \| f(s) - f(t) \| = 0$$

- ***Def.*** *strongly differentiable at* $s \in J$*:*

$$\exists\, g \in \mathcal{H}: \quad \lim_{\tau \to 0} \left\| \frac{1}{\tau} [f(s+\tau) - f(s)] - g \right\| = 0$$

- ***Def.*** *strongly differentiable in* J*:* f is differentiable at each point $s \in J$

$$\exists\, f' : J \to \mathcal{H}: \quad \lim_{\tau \to 0} \left\| \frac{1}{\tau} [f(s+\tau) - f(s)] - f'(s) \right\| = 0 \quad \forall s \in J$$

$$\text{(strong) } derivative\ f'(s) = \frac{d}{ds} f(s) = s - \lim_{\tau \to 0} \tau^{-1} [f(s+\tau) - f(s)]$$

Riemann integral

- Interval: $J = (a, b]$;
- Partition on J: $\Pi = \{s_0, s_1, \dots, s_N; u_1, u_2, \dots, u_N\}$, with $a = s_0 < s_1 < \cdots < s_N = b$ and with $u_k \in (s_{k-1}, s_k]$
- Set $|\Pi| = \max_{k=1,\dots,N} |s_k - s_{k-1}|$
- Set $\sum(\Pi, f) = \sum_{k=1}^{N} (s_k - s_{k-1}) f(u_k)$
- Choose a sequence $\{\Pi_r\}_{n\in\mathbb{N}}$ of partitions of J such that $\lim_{r\to\infty} |\Pi_r| = 0$

Def. *Riemann integral:*

$$\int_J f(s)ds \equiv \int_a^b f(s)ds = s - \lim_{r\to\infty} \sum (\Pi_r, f)$$

Subspaces and dual of a Hilbert space

Definitions and examples for subspaces

- ***Def.*** *Linear manifold $\mathcal{E}$:*

  A linear subset of $\mathcal{H}$, i. e. such that if $f, g \in \mathcal{E}$ and $\alpha \in \mathbb{C}$ then $f + \alpha g \in \mathcal{E}$

- ***Def.*** *Subspace of $\mathcal{H}$:*

  A linear manifold $\mathcal{E}$ which is also complete.

- *Examples of subspaces:*
  - *0*-D: $\{\mathbf{0}\}$
  - *1*-D: $\{\alpha f \,|\, \alpha \in \mathbb{C}\}$
  - ∞-D: all linear combinations of the vectors $e_2, e_4, \ldots$ of an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$

Orthogonality

- ***Def.*** $f, g \in \mathcal{H}$ are *orthogonal:* $\langle f, g \rangle = 0$ *($f \perp g$)*

- ***Def.*** *orthogonal complement* of a subset $\mathcal{N} \subseteq \mathcal{H}$*:* *($\mathcal{N}^{\perp}$)*

  $\mathcal{N}^{\perp} = \{f \in \mathcal{H} | \langle f, g \rangle = 0 \;\; \forall \; g \in \mathcal{N}\}$

- ***Proposition 1.7.*** Projection Theorem

  Let $\mathcal{M}$ be a subspace of $\mathcal{H}$, $\mathcal{M}^{\perp}$ its orthogonal complement.
  Then each vector $f \in \mathcal{H}$ admits a unique decomposition into a component in $\mathcal{M}$ and a component in $\mathcal{M}^{\perp}$:

  $$f = f_1 + f_2 \quad \text{with} \quad f_1 \in \mathcal{M},\, f_2 \in \mathcal{M}^{\perp}.$$

- *Consequence:*

  $\mathcal{H}$ may be viewed as the orthogonal (direct) sum of mutually orthogonal subspaces.

Def. Dual of a Hilbert space $\mathcal{H}^*$

The set of all mappings $\varphi: \mathcal{H} \to \mathbb{C}$ having the following properties:

- *D1.* Linearity: $\varphi(f + \alpha g) = \varphi(f) + \alpha\varphi(g),\ \forall f, g \in \mathcal{H},\ \alpha \in \mathbb{C}$
- *D2.* Boundedness: $|\varphi(f)| \leq c||f||,\ \forall\ f \in \mathcal{H}$

- $\mathcal{H}^*$ is a complex linear vector space
- Normed: $||\varphi|| \equiv ||\varphi||_{\mathcal{H}^*} = \sup_{\mathbf{0} \neq f \in \mathcal{H}} \frac{|\varphi(f)|}{||f||}$

Dual correspondence

- Each vector $g \in \mathcal{H}$ determines an element $\varphi_g \in \mathcal{H}^*$ by setting:

$$\varphi_g(f) = \langle g, f \rangle, f \in \mathcal{H}$$

- The converse is also true!

Proposition 1.8. *Riesz Lemma*

Let $\mathcal{H}$ be a Hilbert space and φ *a bounded linear functional on* $\mathcal{H}$. Then there exists a *unique* vector g in $\mathcal{H}$ such that:

$$\varphi(f) = \langle g, f \rangle \quad \forall \quad f \in \mathcal{H}.$$

- *Consequence:*

$$||\varphi||_{\mathcal{H}^*} = ||g||_{\mathcal{H}} \equiv ||g||$$

- The identification of $\mathcal{H}^*$ with $\mathcal{H}$ is *anti-linear!*

$$\varphi_1(f) = \langle g_1, f \rangle, \varphi_2(f) = \langle g_2, f \rangle \; \forall \; f \in \mathcal{H}:$$

$$(\varphi_1 + \alpha\varphi_2)(f) = \langle g_1 + \bar{\alpha} g_2, f \rangle$$

Measure theory and integration in the sense of Lebesgue

σ-algebra and measure space

- ***Def.*** *σ-algebra:*

  A collection $\mathcal{A}$ of subsets of a set $\mathcal{O}$ satisfying:

  (1) $\phi \in \mathcal{A}, \quad \mathcal{O} \in \mathcal{A}$

  (2) $\mathcal{V} \in \mathcal{A} \;\Leftrightarrow \mathcal{O} \backslash \mathcal{V} \;\in \mathcal{A}$

  (3) $\mathcal{V}_k \in \mathcal{A}$ for $(k = 1,2, \dots) \Rightarrow \cup_k \mathcal{V}_k \in \mathcal{A}$

- ***Def.*** *Measurable space:* $(\mathcal{O}, \mathcal{A})$

- ***Def.*** *Measure* m on $(\mathcal{O}, \mathcal{A})$*:*

  $m : \mathcal{A} \to [0, \infty]$ that is σ-additive

  (i. e. $m(\cup_k \mathcal{V}_k) = \sum_k m(\mathcal{V}_k)$ for each countable family $\{\mathcal{V}_k\}$ of disjoint elements of $\mathcal{A}$, and $m(\phi) = 0$)

- ***Def.*** *Measure space:* $(\mathcal{O}, \mathcal{A}, m)$

Measurable functions

- ***Def.*** *Measurable function* $\varphi : \mathcal{O} \to \mathbb{R}$:

$$\forall\, J \subset \mathbb{R}: \quad \varphi^{-1}(J) := \{x \in \mathcal{O} \mid \varphi(x) \in J\} \in \mathcal{A}$$

- ***Def.*** *Simple function*

$$\varphi = \sum_{k=1}^{N} \alpha_k \chi_{\mathcal{V}_k} \text{ with } \alpha_k \in \mathbb{R}, N < \infty$$

*characteristic function $\chi_{\mathcal{V}_k}(x) = \begin{cases} 1, & x \in \mathcal{V}_k \\ 0, & x \notin \mathcal{V}_k \end{cases}$

  - Measurable if $\mathcal{V}_k \in \mathcal{A}\ \forall k = 1, 2, \dots, N$

- ***Proposition.***

If φ is a measurable function, there exists a sequence $\{\varphi_n\}$ of measurable simple functions such that $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)\ \forall x \in \mathcal{O}$.

If $\varphi \geq 0$, the sequence $\{\varphi_n\}$ can be chosen non-decreasing.

Defining the integral in the sense of Lebesgue

- Integral of a *simple m-integrable function* φ with respect to the measure m on $(\mathcal{O}, \mathcal{A})$ $(m(\mathcal{V}_k) < \infty)$:

$$\int_{\mathcal{O}} \varphi(x)\, m(dx) = \sum_{k=1}^{N} \alpha_k m(\mathcal{V}_k) \ \in \ \mathbb{R}$$

- Integral of a positive measurable function:

$$\int_{\mathcal{O}} \varphi(x)\, m(dx) = \lim_{n\to\infty} \int_{\mathcal{O}} \varphi_n(x) m(dx) < \infty,$$

where $\{\varphi_n\}$ is such that $0 \le \varphi_n(x) \le \varphi(x)$ and $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)\, \forall x \in \mathcal{O}$

- Integral of a general measurable function:

$$\int_{\mathcal{O}} \varphi(x)\, m(dx) = \int_{\mathcal{O}} \varphi_+(x)\, m(dx) - \int_{\mathcal{O}} \varphi_-(x)\, m(dx)$$

- $\varphi : \mathcal{O} \to \mathbb{C}$ is m-integrable: its real part and its imaginary part are m-integrable

Borel σ-algebra

- ***Def.*** *The σ-algebra generated by $\mathcal{B}$*:

  The smallest *σ-algebra $\mathcal{A}$ containing an arbitrary collection $\mathcal{B}$ of subsets of a set $\mathcal{O}$.*

- ***Def.*** *Borel σ-algebra of $\mathcal{O} = \mathbb{R}$:* $\mathcal{A}_{\mathrm{B}}$

  The σ-algebra generated by the collection of all half-open intervals $\mathcal{B} = \{(a, b] \mid -\infty < a \leq b < +\infty\}$

- ***Def.*** *Borel measure of $\mathbb{R}$:*

  Extension of the mapping

  $$m: \mathcal{B} \to [0, \infty);\ m\big((a, b]\big) = \ell\big((a, b]\big) \equiv b - a$$

  to a measure on $(\mathbb{R}, \mathcal{A}_{\mathrm{B}})$.

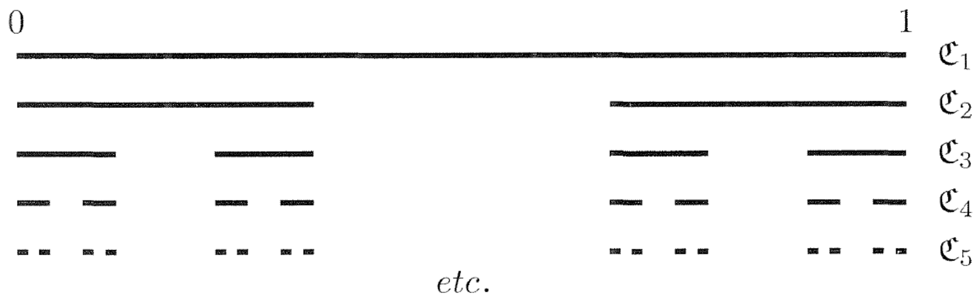
Borel sets: the elements of $\mathcal{A}_\mathrm{B}$

- The length of a Borel set $\mathcal{V}$?
  - Consider covers of $\mathcal{V}$, such that $\mathcal{V} \subseteq \bigcup_k J_k$ with $\{J_k\}_{k\in\mathbb{N}}$ countable collection of half-open intervals, where $J_k = (a_k, b_k],\ a_k < b_k$.
  - Define:

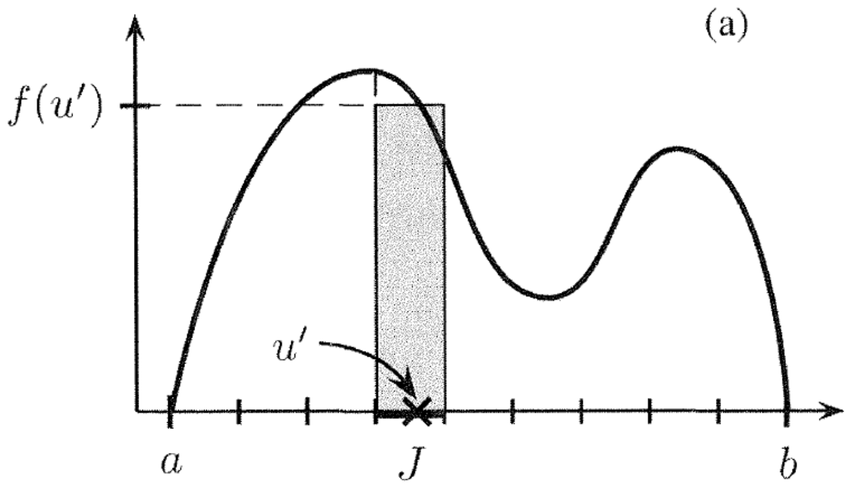
$$\ell(\mathcal{V}) = \inf \sum_k \ell(J_k) \equiv \inf \sum_k (b_k - a_k)$$

$$\Rightarrow m_\mathrm{B}\,(\mathcal{V}) = \ell(\mathcal{V}) \text{ for } \mathcal{V} \in \mathcal{A}_\mathrm{B}$$

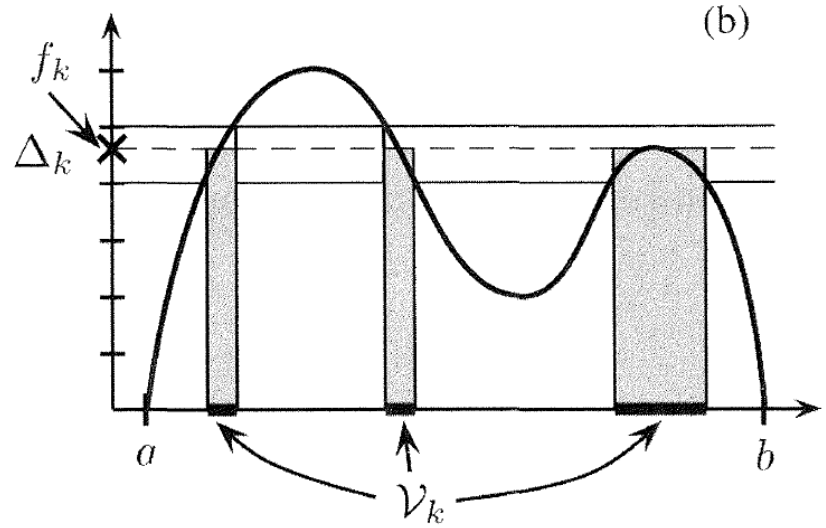
- Examples of Borel sets of length zero:
  - Finite: $\{x_0\}$
  - Countable: $\{x_1, x_2, \dots\}$
  - Uncountable: Cantor set $\mathcal{C}$



Riemann integration VS. Lebesgue integration



Partitions of the domain of f



Partitions of the range of f

Lebesgue σ-algebra $\mathcal{A}_{\mathrm{L}}$

- Extension of the Borel measure to σ-algebras larger than $\mathcal{A}_{\mathrm{B}}$

- Elements $\mathcal{V} \in \mathcal{A}_{\mathrm{L}}$:

$$\mathcal{V} = \mathcal{W} \cup \mathcal{V}_0 \ \text{, where } \mathcal{W} \in \mathcal{A}_{\mathrm{B}},$$

$$\mathcal{V}_0 : \exists\, \mathcal{U} \in \mathcal{A}_{\mathrm{B}} : \ell(\mathcal{U}) = 0 \ \text{ and } \ \mathcal{V}_0 \subset \mathcal{U}$$

- Lebesgue measure:

$$m_{\mathrm{L}}(\mathcal{V}) = m_{\mathrm{B}}(\mathcal{W}) = \ell(\mathcal{W})$$

$$(\textbf{Notation}\text{: } m_{\mathrm{L}}(dx) = m_{\mathrm{B}}(dx) = dx)$$

- $(\mathbb{R}, \mathcal{A}_L, m_L)$ *is complete:*

  $\mathcal{A}_{\mathrm{L}}$ contains all subsets of sets of measure zero

- In $\mathbb{R}^n$: analogue

L^p spaces

Definition

Given $(\mathcal{O}, \mathcal{A}, m)$ a general measure space.

- Two measurable functions φ_1 and φ_2 are said to be ***equivalent***:
  - If they are equal m-almost everywhere (differ at most on a null set with respect to the measure m)
  - If and only if $m(\mathcal{W}) = 0$, where $\mathcal{W} = \{x \in \mathcal{O} \mid \varphi_1(x) \neq \varphi_2(x)\} \in \mathcal{A}$

- If φ_1 and φ_2 are equivalent:
  - $\forall\, t > 0 : \int |\varphi_1 - \varphi_2|^t \, m(dx) = 0$
  - If one of them is m-integrable: $\int_{\mathcal{O}} \varphi_1(x) m(dx) = \int_{\mathcal{O}} \varphi_2(x) m(dx)$

Def. *Normed vector space* $L^p(\mathcal{O}, m) \equiv L^p(\mathcal{A}, \mathcal{A}, m)$ (with $1 \leq p < \infty$)*:*

the set of equivalence classes of measurable functions $f : \mathcal{O} \to \mathbb{C}$, such that $||f||_p < \infty$, where the ***norm*** is:

$$\|f\|_p := \left[\int_{\mathcal{O}} |f(x)|^P m(dx) \right]^{1/p}, \text{where } p \in [1, +\infty)$$

Properties

- $L^p(\mathcal{O}, m)$ *is a Banach space* (complete normed vector space).

- $C_0^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.
  (***Def.*** *$\mathcal{D}$ Dense subset of $\mathcal{H}$:*
  $$\forall\, f \in \mathcal{H}, \epsilon > 0 \colon \exists g \in \mathcal{D} \text{ such that } \left|\left|f - g\right|\right| < \epsilon)$$

Dominated convergence theorem

Proposition 1.9.

Assume that:

a. $f_n, h \in L^1(\mathcal{O}, m)$,
b. for each $n \in \mathbb{N}$: $|f_n(x)| \leq h(x)$ m-almost everywhere,
c. $\lim_{n\to\infty} f_n(x) = f(x)$ m-almost everywhere.

Then the limit function f is m-integrable and one has:

$$\lim_{n\to\infty} \int_{\mathcal{O}} f_n(x) m(dx) = \int_{\mathcal{O}} f(x) m(dx)$$

Some inclusion relations:

$L^p(\mathcal{O}, m) \cap L^q(\mathcal{O}, m) \subseteq L^r(\mathcal{O}, m)$ if $1 \leq p \leq r \leq q \leq \infty$

$f \in L^q(\mathcal{O}, m)$ and $m(\mathcal{V}) < \infty \Rightarrow \chi_{\mathcal{V}} f \in L^r(\mathcal{O}, m) \quad \forall r \in [1, q]$

$L^2(\mathcal{O}, m)$ space

- Hilbert space with scalar product:

$$\langle f, g \rangle = \int_{\mathcal{O}} \overline{f(x)} \cdot g(x)\, m(dx)$$

- Generalization: consider functions with values in a Hilbert space $\mathcal{K}$

$\Rightarrow$ $L^2(\mathcal{O}, \mathcal{K}, m)$ with scalar product:

$$\langle f, g \rangle = \int_{\mathcal{O}} \langle f(x), g(x) \rangle_{\mathcal{K}}\, m(dx)$$

and norm:

$$\left[\int_{\mathcal{O}} \|f(x)\|_{\mathcal{K}}^2\, m(dx) \right]^{1/2}$$

Thank you for your attention!