1. HILBERT SPACES

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Definition and elementary properties of Hilbert spaces
Def. Hilbert space $\mathcal{H}$

H1. $\mathcal{H}$ complex linear vector space.
   - Standard properties for vector addition and multiplication with a scalar
   - Unique zero vector $0 \in \mathcal{H}$

H2. $\mathcal{H}$ is equipped with a strictly positive scalar product.
   - $\langle g, f \rangle = \overline{\langle f, g \rangle}$
   - $\langle f, g + \alpha h \rangle = \langle f, g \rangle + \alpha \langle f, h \rangle$ \(\Rightarrow\) **norm**: $\|f\| := [\langle f, f \rangle]^{1/2}$
   - $\langle f, f \rangle > 0$ except for $f = 0$

H3. $\mathcal{H}$ is complete.
   - Each Cauchy sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$ has a limit in $\mathcal{H}$
   - $\{f_n\}_{n \in \mathbb{N}} \in \mathcal{H}: \lim_{m,n \to \infty} \|f_n - f_m\| = 0 \Rightarrow \exists f \in \mathcal{H}: \lim_{n \to \infty} \|f - f_m\| = 0$

H4. $\mathcal{H}$ is separable.
   - $\mathcal{H}$ has a countable orthonormal basis
   \(\Rightarrow\) **dimension**: number of elements in the basis, independent of chosen basis
Elements of $\mathcal{H}$: vectors

Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of $\mathcal{H}$, $f \in \mathcal{H}$ arbitrary. Then there exists a sequence $\{\alpha_1, \alpha_2, \ldots\}$ of complex numbers such that:

$$f = \sum_k \alpha_k e_k \quad \text{and} \quad |f|^2 = \sum_k |\alpha_k|^2$$

with coefficients $\alpha_k = \langle e_k, f \rangle$

- finite dimensionality: as in linear algebra, $k = 1, N; \ N < \infty$
- infinite-dimensional spaces: convergence in the sense that:

$$\left\| f - \sum_{k=1}^N \alpha_k e_k \right\| \to 0 \quad \text{as} \quad N \to \infty$$
Elementary properties of $\mathcal{H}$

- Polarisation identity:
  \[ 4\langle f, g \rangle = \|f + g\|^2 - \|f - g\|^2 - i\|f + ig\|^2 + i\|f - ig\|^2 \]

- Schwarz inequality:
  \[ \langle f, g \rangle \leq \|f\| \cdot \|g\| \]

- Triangle inequality:
  \[ \|f + g\| \leq \|f\| + \|g\| \]
  \[ \|f + g\|^2 \leq 2\|f\|^2 + 2\|g\|^2 \]
  \[ \|\|f\| - \|g\|| \leq \|f - g\| \]
Convergence: weak, strong
Types of convergence

- **Def. Strong convergence:** \( s\text{-}\lim_{n \to \infty} f_n = f \)
  
  A sequence of vectors \( \{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{H} \) is said to converge strongly to a vector \( f \in \mathcal{H} \) if \( \|f_n - f\| \to 0 \) \( (n \to \infty) \)

- **Def. Weak convergence:** \( w\text{-}\lim_{n \to \infty} f_n = f \)
  
  A sequence of vectors \( \{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{H} \) is said to converge weakly to a vector \( f \in \mathcal{H} \) if, for each \( g \in \mathcal{H} \), the sequence of complex numbers \( \{\langle f_n, g \rangle\}_{n \in \mathbb{N}} \to \langle f, g \rangle \) \( (n \to \infty) \)
**Proposition 1.1.**

- a. Let \( \{ f_n \}_{n \in \mathbb{N}} \) be a sequence of vectors in \( \mathcal{H} \). Then:
  \[
  s\lim_{n \to \infty} f_n = f \iff w\lim_{n \to \infty} f_n = f \quad \text{and} \quad \lim_{n \to \infty} \|f_n\| = \|f\|.
  \]

- b. If \( s\lim_{n \to \infty} f_n = f \) and \( s\lim_{n \to \infty} g_n = g \), then \( \lim_{n \to \infty} \langle f_n, g_n \rangle = \langle f, g \rangle \)

**Example 1.2.**

Every infinite orthonormal sequence \( \{ e_n \} \) converges weakly to \( 0 \), but never converges strongly.
Vector - valued functions
Definitions

- **Def. vector-valued function**: a mapping \( f : \Lambda \rightarrow \mathcal{H} \)
  - \( \Lambda \) is an interval \( J \): \( f(s) \) for \( s \in J \)

- **Def.** \( f : J \rightarrow \mathcal{H} \) strongly continuous:
  \[
  \forall t \in J: \lim_{s \to t, s \in J} \| f(s) - f(t) \| = 0
  \]

- **Def.** strongly differentiable at \( s \in J \):
  \[
  \exists g \in \mathcal{H}: \lim_{\tau \to 0} \left\| \frac{1}{\tau} [f(s + \tau) - f(s)] - g \right\| = 0
  \]

- **Def.** strongly differentiable in \( J \): \( f \) is differentiable at each point \( s \in J \)
  \[
  \exists f' : J \rightarrow \mathcal{H}: \lim_{\tau \to 0} \left\| \frac{1}{\tau} [f(s + \tau) - f(s)] - f'(s) \right\| = 0 \quad \forall s \in J
  \]
  
  (strong) derivative \( f'(s) = \frac{d}{ds} f(s) = s - \lim_{\tau \to 0} \tau^{-1}[f(s + \tau) - f(s)] \)
Riemann integral

- Interval: $J = (a, b]$;
- Partition on $J$: $\Pi = \{s_0, s_1, \ldots, s_N; u_1, u_2, \ldots, u_N\}$, with $a = s_0 < s_1 < \ldots < s_N = b$ and with $u_k \in (s_{k-1}, s_k]$.
- Set $|\Pi| = \max_{k=1,\ldots,N} |s_k - s_{k-1}|$.
- Set $\sum(\Pi, f) = \sum_{k=1}^{N} (s_k - s_{k-1})f(u_k)$.
- Choose a sequence $\{\Pi_r\}_{n \in \mathbb{N}}$ of partitions of $J$ such that $\lim_{r \to \infty} |\Pi_r| = 0$.

**Def. Riemann integral:**

$$\int_{J} f(s) ds \equiv \int_{a}^{b} f(s) ds = s - \lim_{r \to \infty} \sum (\Pi_r, f)$$
Subspaces and dual of a Hilbert space
Definitions and examples for subspaces

**Def. Linear manifold $\mathcal{E}$:**
A linear subset of $\mathcal{H}$, i.e. such that if $f, g \in \mathcal{E}$ and $\alpha \in \mathbb{C}$ then $f + \alpha g \in \mathcal{E}$

**Def. Subspace of $\mathcal{H}$:**
A linear manifold $\mathcal{E}$ which is also complete.

**Examples of subspaces:**
- 0-D: $\{0\}$
- 1-D: $\{\alpha f \mid \alpha \in \mathbb{C}\}$
- $\infty$-D: all linear combinations of the vectors $e_2, e_4, \ldots$ of an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$
Orthogonality

- **Def.** $f, g \in \mathcal{H}$ are orthogonal: $\langle f, g \rangle = 0$ \hspace{1cm} ($f \perp g$)

- **Def.** orthogonal complement of a subset $\mathcal{N} \subseteq \mathcal{H}$: \hspace{1cm} ($\mathcal{N}^\perp$)
  \[ \mathcal{N}^\perp = \{ f \in \mathcal{H} | \langle f, g \rangle = 0 \ \forall \ g \in \mathcal{N} \} \]

- **Proposition 1.7.** Projection Theorem
  Let $\mathcal{M}$ be a subspace of $\mathcal{H}$, $\mathcal{M}^\perp$ its orthogonal complement. Then each vector $f \in \mathcal{H}$ admits a unique decomposition into a component in $\mathcal{M}$ and a component in $\mathcal{M}^\perp$:
  \[ f = f_1 + f_2 \ \text{with} \ f_1 \in \mathcal{M}, \ f_2 \in \mathcal{M}^\perp. \]

- **Consequence:**
  $\mathcal{H}$ may be viewed as the orthogonal (direct) sum of mutually orthogonal subspaces.
Def. Dual of a Hilbert space $\mathcal{H}^*$

The set of all mappings $\varphi: \mathcal{H} \rightarrow \mathbb{C}$ having the following properties:

- **D1. Linearity:** $\varphi(f + \alpha g) = \varphi(f) + \alpha \varphi(g)$, $\forall f, g \in \mathcal{H}$, $\alpha \in \mathbb{C}$
- **D2. Boundedness:** $|\varphi(f)| \leq c||f||$, $\forall f \in \mathcal{H}$

- $\mathcal{H}^*$ is a complex linear vector space
- Normed: $||\varphi|| \equiv ||\varphi||_{\mathcal{H}^*} = \sup_{0 \neq f \in \mathcal{H}} \frac{|\varphi(f)|}{||f||}$
Each vector $g \in \mathcal{H}$ determines an element $\varphi_g \in \mathcal{H}^*$ by setting:

$$\varphi_g(f) = \langle g, f \rangle, f \in \mathcal{H}.$$ 

The converse is also true!

**Proposition 1.8. Riesz Lemma**

Let $\mathcal{H}$ be a Hilbert space and $\varphi$ a bounded linear functional on $\mathcal{H}$. Then there exists a unique vector $g$ in $\mathcal{H}$ such that:

$$\varphi(f) = \langle g, f \rangle \quad \forall \quad f \in \mathcal{H}.$$ 

**Consequence:**

$$||\varphi||_{\mathcal{H}^*} = ||g||_{\mathcal{H}} \equiv ||g||$$

The identification of $\mathcal{H}^*$ with $\mathcal{H}$ is anti-linear!

$$\varphi_1(f) = \langle g_1, f \rangle, \varphi_2(f) = \langle g_2, f \rangle \forall f \in \mathcal{H}:$$

$$(\varphi_1 + \alpha \varphi_2)(f) = \langle g_1 + \bar{\alpha}g_2, f \rangle$$
Measure theory and integration in the sense of Lebesgue
**σ-algebra and measure space**

- **Def. σ-algebra:**
  A collection $\mathcal{A}$ of subsets of a set $\mathcal{O}$ satisfying:
  - (1) $\phi \in \mathcal{A}$, $\mathcal{O} \in \mathcal{A}$
  - (2) $\mathcal{V} \in \mathcal{A} \Leftrightarrow \mathcal{O}\setminus\mathcal{V} \in \mathcal{A}$
  - (3) $\mathcal{V}_k \in \mathcal{A}$ for ($k = 1, 2, \ldots$) $\Rightarrow \bigcup_k \mathcal{V}_k \in \mathcal{A}$

- **Def. Measurable space:** $(\mathcal{O}, \mathcal{A})$

- **Def. Measure $m$ on $(\mathcal{O}, \mathcal{A})$:**
  $m : \mathcal{A} \rightarrow [0, \infty]$ that is $\sigma$-additive
  - (i.e. $m(\bigcup_k \mathcal{V}_k) = \sum_k m(\mathcal{V}_k)$ for each countable family $\{\mathcal{V}_k\}$ of disjoint elements of $\mathcal{A}$, and $m(\phi) = 0$)

- **Def. Measure space:** $(\mathcal{O}, \mathcal{A}, m)$
Measurable functions

**Def.** Measurable function \( \varphi : \mathcal{O} \to \mathbb{R} : \)

\[
\forall J \subset \mathbb{R}: \quad \varphi^{-1}(J) := \{x \in \mathcal{O} \mid \varphi(x) \in J \} \in \mathcal{A}
\]

**Def.** Simple function

\[
\varphi = \sum_{k=1}^{N} \alpha_k \chi_{\mathcal{V}_k} \quad \text{with} \quad \alpha_k \in \mathbb{R}, N < \infty
\]

*characteristic function \( \chi_{\mathcal{V}_k}(x) = \begin{cases} 
1, & x \in \mathcal{V}_k \\
0, & x \notin \mathcal{V}_k
\end{cases} \)

Measurable if \( \mathcal{V}_k \in \mathcal{A} \forall k = 1, 2, \ldots, N \)

**Proposition.**

If \( \varphi \) is a measurable function, there exists a sequence \( \{\varphi_n\} \) of measurable simple functions such that \( \varphi(x) = \lim_{n \to \infty} \varphi_n(x) \forall x \in \mathcal{O} \).

If \( \varphi \geq 0 \), the sequence \( \{\varphi_n\} \) can be chosen non-decreasing.
Defining the integral in the sense of Lebesgue

- Integral of a \textit{simple m-integrable function} \( \varphi \) with respect to the measure \( m \) on \((\mathcal{O}, \mathcal{A})\) \( (m(V_k) < \infty) \):

\[
\int_{\mathcal{O}} \varphi(x) \, m(dx) = \sum_{k=1}^{N} \alpha_k m(V_k) \in \mathbb{R}
\]

- Integral of a positive measurable function:

\[
\int_{\mathcal{O}} \varphi(x) \, m(dx) = \lim_{n \to \infty} \int_{\mathcal{O}} \varphi_n(x)m(dx) < \infty,
\]

where \( \{\varphi_n\} \) is such that \( 0 \leq \varphi_n(x) \leq \varphi(x) \) and

\[
\lim_{n \to \infty} \varphi_n(x) = \varphi(x) \quad \forall x \in \mathcal{O}
\]

- Integral of a general measurable function:

\[
\int_{\mathcal{O}} \varphi(x) \, m(dx) = \int_{\mathcal{O}} \varphi^+(x) \, m(dx) - \int_{\mathcal{O}} \varphi^-(x) \, m(dx)
\]

- \( \varphi : \mathcal{O} \to \mathbb{C} \) is \( m \)-integrable: its real part and its imaginary part are \( m \)-integrable.
Borel $\sigma$-algebra

Def. The $\sigma$-algebra generated by $\mathcal{B}$:
The smallest $\sigma$-algebra $\mathcal{A}$ containing an arbitrary collection $\mathcal{B}$ of subsets of a set $\mathcal{O}$.

Def. Borel $\sigma$-algebra of $\mathcal{O} = \mathbb{R}$: $\mathcal{A}_B$
The $\sigma$-algebra generated by the collection of all half-open intervals $\mathcal{B} = \{(a, b] \mid -\infty < a \leq b < +\infty\}$

Def. Borel measure of $\mathbb{R}$:
Extension of the mapping
$m: \mathcal{B} \to [0, \infty); m((a, b]) = \ell((a, b]] \equiv b - a$
to a measure on $(\mathbb{R}, \mathcal{A}_B)$. 
Borel sets: the elements of $\mathcal{A}_B$

- The length of a Borel set $\mathcal{V}$?
  - Consider covers of $\mathcal{V}$, such that $\mathcal{V} \subseteq \bigcup_k J_k$ with $\{J_k\}_{k \in \mathbb{N}}$ countable collection of half-open intervals, where $J_k = (a_k, b_k]$, $a_k < b_k$.
  - Define:
    \[
    \ell(\mathcal{V}) = \inf \sum_k \ell(J_k) \equiv \inf \sum_k (b_k - a_k)
    \]
    \[\Rightarrow m_B(\mathcal{V}) = \ell(\mathcal{V}) \text{ for } \mathcal{V} \in \mathcal{A}_B\]

- Examples of Borel sets of length zero:
  - Finite: $\{x_0\}$
  - Countable: $\{x_1, x_2, \ldots\}$
  - Uncountable: Cantor set $\mathcal{C}$

![Cantor set diagram]
Riemann integration VS. Lebesgue integration

Partitions of the domain of $f$

Partitions of the range of $f$
Lebesgue $\sigma$-algebra $\mathcal{A}_L$

- Extension of the Borel measure to $\sigma$-algebras larger than $\mathcal{A}_B$

- Elements $\mathcal{V} \in \mathcal{A}_L$:

  $$\mathcal{V} = \mathcal{W} \cup \mathcal{V}_0 \text{, where } \mathcal{W} \in \mathcal{A}_B,$$

  $$\mathcal{V}_0 : \exists \mathcal{U} \in \mathcal{A}_B : \ell(\mathcal{U}) = 0 \text{ and } \mathcal{V}_0 \subset \mathcal{U}$$

- Lebesgue measure:

  $$m_L(\mathcal{V}) = m_B(\mathcal{W}) = \ell(\mathcal{W})$$

  \textbf{(Notation:} $m_L(dx) = m_B(dx) = dx$)

- ($\mathbb{R}, \mathcal{A}_L, m_L$) is complete:

  $\mathcal{A}_L$ contains all subsets of sets of measure zero

- In $\mathbb{R}^n$: analogue
$L^p$ spaces
Definition

Given \((\mathcal{O}, \mathcal{A}, m)\) a general measure space.

- Two measurable functions \(\varphi_1\) and \(\varphi_2\) are said to be equivalent:
  - If they are equal \(m\)-almost everywhere (differ at most on a null set with respect to the measure \(m\))
  - If and only if \(m(\mathcal{W}) = 0\), where \(\mathcal{W} = \{x \in \mathcal{O} \mid \varphi_1(x) \neq \varphi_2(x)\}\) \(\in \mathcal{A}\)

- If \(\varphi_1\) and \(\varphi_2\) are equivalent:
  - \(\forall t > 0 : \int |\varphi_1 - \varphi_2|^t m(dx) = 0\)
  - If one of them is \(m\)-integrable: \(\int_\mathcal{O} \varphi_1(x)m(dx) = \int_\mathcal{O} \varphi_2(x)m(dx)\)

**Def. Normed vector space** \(L^p(\mathcal{O}, m) \equiv L^p(\mathcal{O}, \mathcal{A}, m)\) (with \(1 \leq p < \infty\)):

the set of equivalence classes of measurable functions \(f: \mathcal{O} \to \mathbb{C}\), such that \(\|f\|_p < \infty\), where the **norm** is:

\[
\|f\|_p := \left[ \int_\mathcal{O} |f(x)|^p m(dx) \right]^{1/p}, \text{ where } p \in [1, +\infty)
\]
Properties

- $L^p(\mathcal{O}, m)$ is a Banach space (complete normed vector space).

- $C_0^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.
  
  (Def. $D$ Dense subset of $\mathcal{H}$:
  \[ \forall f \in \mathcal{H}, \epsilon > 0: \exists g \in D \text{ such that } \|f - g\| < \epsilon \])
Dominated convergence theorem

**Proposition 1.9.**

Assume that:

a. \( f_n, h \in L^1(\mathcal{O}, m) \),

b. for each \( n \in \mathbb{N} \): \( |f_n(x)| \leq h(x) \) \( m \)-almost everywhere,

c. \( \lim_{n \to \infty} f_n(x) = f(x) \) \( m \)-almost everywhere.

Then the limit function \( f \) is \( m \)-integrable and one has:

\[
\lim_{n \to \infty} \int_{\mathcal{O}} f_n(x) m(dx) = \int_{\mathcal{O}} f(x) m(dx)
\]

Some inclusion relations:

\( L^p(\mathcal{O}, m) \cap L^q(\mathcal{O}, m) \subseteq L^r(\mathcal{O}, m) \) if \( 1 \leq p \leq r \leq q \leq \infty \)

\( f \in L^q(\mathcal{O}, m) \) and \( m(\mathcal{V}) < \infty \) \( \Rightarrow \chi_{\mathcal{V}} f \in L^r(\mathcal{O}, m) \) \( \forall r \in [1, q] \)
$L^2(\mathcal{O}, m)$ space

- Hilbert space with scalar product:

$$\langle f, g \rangle = \int_{\mathcal{O}} \overline{f(x)} \cdot g(x) \, m(dx)$$

- Generalization: consider functions with values in a Hilbert space $\mathcal{K}$

$\Rightarrow \quad L^2(\mathcal{O}, \mathcal{K}, m)$ with scalar product:

$$\langle f, g \rangle = \int_{\mathcal{O}} \langle f(x), g(x) \rangle_{\mathcal{K}} \, m(dx)$$

and norm:

$$\left[ \int_{\mathcal{O}} \| f(x) \|_{\mathcal{K}}^2 \, m(dx) \right]^{1/2}$$
Thank you for your attention!