Summer term 2023
Introduction to Fluid Mechanics
Exercise sheet 1
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Karlsruhe Institute of Technology
JProf. Dr. Xian Liao
M.Sc. Rebekka Zimmermann

Exercise 1

Let $\rho \geq 0$ denote the density, $u \in \mathbb{R}^N$ the velocity vector field, and $e \geq 0$ the (internal) energy of a fluid in $\mathbb{R}^N$, $N = 2, 3$.

1. We define the total energy E of the fluid as $E = \frac{1}{2}\rho|u|^2 + \rho e$. Recall that E satisfies the evolution equation

$$\partial_t E + \mathrm{div}\,(u(E+p)) = \mathrm{div}\,(\tau \cdot u) - \mathrm{div}\,(q) + \rho f \cdot u, \tag{1}$$

where $f = (f^1 \dots f^N)^T$ denotes an external force, p the pressure, and $\tau = (\tau_{ij})_{i,j=1,\dots,N}$ the viscous stress tensor. Show that

$$\partial_t(\rho e) + \mathrm{div}\,(\rho u e) + p\,\mathrm{div}\,u = -\mathrm{div}\,(q) + \tau : d$$

using the evolution equations for the density ρ and the momentum ρu:

$$\begin{aligned}&\partial_t \rho + \mathrm{div}\,(\rho u) = 0,\\ &\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \mathrm{div}\,\tau + \nabla p = \rho f.\end{aligned}$$

2. Let θ be the temperature of the fluid, and let the specific entropy $s = s(\rho, \theta)$ be defined by the equations

$$\frac{\partial s}{\partial \theta} = \frac{1}{\theta}\frac{\partial e}{\partial \theta}, \quad \frac{\partial s}{\partial \rho} = \frac{1}{\theta}\Big(\frac{\partial e}{\partial \rho} - \frac{p}{\rho^2}\Big).$$

Use the evolution equations for the density ρ and the internal energy ρe to show that s satisfies the entropy equation

$$\partial_t(\rho s) + \mathrm{div}\,\Big(\rho u s + \frac{q}{\theta}\Big) = \frac{1}{\theta}\tau : d - \frac{1}{\theta^2} q \cdot \nabla\theta.$$

3. Consider an ideal gas model where the pressure p and the energy e depend on ρ, θ as follows

$$p = (\gamma - 1)\rho e, \quad e = C_v \theta,$$

where $\gamma > 1$ and $C_v > 0$ are constants. The entropy s above then takes the form

$$s = C_v\big(\log(e) + (1-\gamma)\log(\rho)\big).$$

Assume moreover that $\mu = \lambda = \kappa = 0$. The pressure is then given by $p(\rho) = a\rho^\gamma$ for some constant $a > 0$, and the total energy $E = \frac{1}{2}\rho|u|^2 + P(\rho)$, where the function $P : \mathbb{R} \to \mathbb{R}$ satisfies

$$P'(z)z - P(z) = p(z), \quad z \in \mathbb{R}.$$

Deduce the energy equation (1) from the evolution equation for the density ρ and for the momentum ρu.

Exercise 2

Let $u = u(t,x) \in L^1_{\text{loc}}(\mathbb{R}; \text{Lip}(\mathbb{R}^N, \mathbb{R}^N))$ be the velocity vector field of a fluid in R^N, $N = 2, 3$. For $t \in \mathbb{R}$ let $X(t, \cdot) = X_t : \mathbb{R}^N \to \mathbb{R}^N$ be the trajectory

$$\begin{cases} \partial_t X(t,y) = u(t, X(t,y)), \\ X(t,y)|_{t=0} = y. \end{cases}$$

1. We define the Jacobian $J(t,y) = \det(\nabla_y X_t(y))$ for $(t,y) \in \mathbb{R} \times \mathbb{R}^N$. Show that J satisfies

   $$\partial_t J(t,y) = \operatorname{div} u(t, X(t,y)) J(t,y).$$

2. Let $\rho_0 \in L^\infty(\mathbb{R}^N; [0,\infty))$ and suppose the fluid is incompressible, i.e. $\operatorname{div} u = 0$. Show that the equation

   $$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(0, \cdot) = \rho_0 \end{cases}$$

   admits a unique solution which is given by

   $$\rho(t, X(t,y)) = \rho_0(y), \quad (t,y) \in \mathbb{R} \times \mathbb{R}^N.$$

In the following let $N = 3$ and let u solve the classical incompressible Euler equation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0. \end{cases}$$

We set $U = \nabla u$, and denote by $d = \frac{1}{2}(U + U^T)$ and $a = \frac{1}{2}(U - U^T)$ its symmetric and antisymmetric part, respectively.

3. Recall that the symmetric part d of ∇u satisfies the evolution equation

   $$\partial_t d + u \cdot \nabla d + d^2 + a^2 + \nabla^2 p = 0.$$

   Deduce from this the following Poisson equation for the pressure p:

   $$-\Delta p = tr(\nabla u)^2.$$

4. Recall that a satisfies

   $$\partial_t a + u \cdot \nabla a + da + ad = 0.$$

   Show that this equation is equivalent to the vorticity $\omega = \operatorname{curl}(u)$ satisfying

   $$\partial_t \omega + u \cdot \nabla \omega = d\omega.$$

5. Derive also the equation for the vorticity $\omega = \operatorname{curl}(u)$:

   $$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \\ \omega(0, \cdot) = \omega_0. \end{cases}$$

   Show that this equation has a unique solution given by

   $$\omega(t, X(t,y)) = \omega_0(y) \cdot \nabla_y X(t,y).$$