Summary term 2023

Karlsruhe Institute of Technology

Introduction to Fluid Mechanics

Exercise sheet 2

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Exercise 3

1. Let $d = d(t) \in \mathbb{R}^{3\times 3}$ be a symmetric and trace-free matrix, and let $\omega = \omega(t)$ be determined by the ODE equation on $\mathbb{R}^3$:

$$\frac{d}{dt}\omega = d\omega, \quad \omega|_{t=0} = \omega_0,$$

where $\omega_0 \in \mathbb{R}^3$. Show that $u := \frac{1}{2}\omega \times x + d\,x$ satisfies

(i) $\operatorname{div} u = 0$,

(ii) $\operatorname{curl} u = \omega$,

(iii) $\frac{1}{2}(\nabla u + \nabla^T u) = d$,

(iv) $ah := \frac{1}{2}(\nabla u - \nabla^T u)h = \frac{1}{2}\omega \times h$, $\forall h \in \mathbb{R}^3$,

(v) $\partial_t d + u \cdot \nabla d + d^2 + a^2 + \nabla^2 \Pi = 0$, for $\Pi = -\frac{1}{2}\big(\partial_t d + d^2 + a^2\big)x \cdot x$.

2. Let $\gamma_1, \gamma_2 > 0$, and let the vector field $u : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$u(t,x) = \begin{pmatrix} -\gamma_1 x_1 - \frac{1}{2}e^{(\gamma_1+\gamma_2)t}\alpha x_2 \\ -\gamma_2 x_2 + \frac{1}{2}e^{(\gamma_1+\gamma_2)t}\alpha x_1 \\ (\gamma_1+\gamma_2)x_3 \end{pmatrix}.$$

Determine the function $\Pi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that the incompressible Euler equation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = 0, \\ \operatorname{div} u = 0 \end{cases}$$

is satisfied.

Exercise 4

1. Let $N \in \mathbb{N}$, $N \geq 2$. For $x \in \mathbb{R}^N \setminus \{0\}$ the fundamental solution to the Laplace-equation is given by

$$\Gamma(x) = \begin{cases} -\frac{1}{2\pi}\ln|x|, & N = 2, \\ \frac{1}{(N-2)c_N}|x|^{-(N-2)}, & N \geq 3, \end{cases}$$

where $c_N = |\partial B_1(0)|$ denotes the volume of the unit sphere in $\mathbb{R}^N$. Show that

$$\partial_{x_j}\Gamma(x) = -\frac{1}{c_N}\frac{x_j}{|x|^N},$$
$$\partial_{x_i x_j}\Gamma(x) = -\frac{1}{c_N}\Big(\frac{1}{|x|^N}\delta_{ij} - N\frac{x_i x_j}{|x|^{N+2}}\Big),$$
$$\Delta\Gamma(x) = 0,$$

for $x \in \mathbb{R}^N \setminus \{0\}$ and $i, j \in \{1, ..., N\}$.

2. Show that

   (i) Γ, $\partial_{x_j}\Gamma \in L^1_{\text{loc}}(\mathbb{R}^N)$,

   (ii) $\partial_{x_i x_j}\Gamma \in L^1_{\text{loc}}(\mathbb{R}^N \backslash \{0\})$

   for $i, j \in \{1, ..., N\}$.

3. Let $f \in L^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$, and suppose that $\int_{|x|\geq 1} |f(x)| \ln |x| dx < \infty$ if $N = 2$. Recall that $v := \Gamma * f$ satisfies

$$\partial_{x_i x_j} v(x) = \text{p.v.} \int_{\mathbb{R}^N} g_{ij}(x-y) f(y) dy - \frac{1}{N} f(x) \delta_{ij},$$

   for $x \in \mathbb{R}^N$ and $i, j \in \{1, ..., N\}$. Deduce that $v \in C^2(\mathbb{R}^N)$ with $-\Delta v = f$.