Summary term 2023

Karlsruhe Institute of Technology

Introduction to Fluid Mechanics

Exercise sheet 3

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JProf. Dr. Xian Liao

M.Sc. Rebekka Zimmermann

Exercise 5

Let $\omega : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth and decaying sufficiently fast at infinity. Let $\tilde{u} = -K_3 * \omega$, where $K_3(x) \in \mathbb{R}^{3\times 3}$ is the matrix such that

$$K_3(x)h = \frac{1}{4\pi}\frac{x \times h}{|x|^3}, \quad \forall h \in \mathbb{R}^3.$$

1. Show that $\tilde{u}$ satisfies $-\Delta \tilde{u} = \nabla \times \omega$.
2. Show that if $\omega \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then $\tilde{u} \in L^r(\mathbb{R}^3)$ for $r \in (\frac{3}{2}, \infty]$.

Exercise 6

Let $N \in \mathbb{N}$, $N \geq 2$. For sufficiently smooth functions v and w we set

$$A(v, w) = \nabla\Gamma * \mathrm{tr}(\nabla v \nabla w) = -\frac{1}{4\pi}\frac{x}{|x|^3} * (\sum_{i,j} \partial_i v^j \partial_j w^i).$$

Show that u satisfies

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = 0, \\ \mathrm{div}\, u = 0, \end{cases}$$

if and only if it satisfies

$$\begin{cases} \partial_t u + u \cdot \nabla u + A(u, u) = 0, \\ u|_{t=0} = u_0, \quad \mathrm{div}\, u_0 = 0. \end{cases}$$

Exercise 7

Let $\alpha \in (0, 1)$.

1. Show that $C^{k,\alpha} := C^{k,\alpha}(\mathbb{R}^N)$ is a Banach space for $k \in \mathbb{N} \cup \{0\}$.
2. Show that there exists a constant C such that for any $f, g \in C^{1,\alpha}$,

   (i) $\|fg\|_{C^{k,\alpha}} \leq C\|f\|_{C^{k,\alpha}}\|g\|_{C^{k,\alpha}}$, for $k = 0, 1$,

   (ii) $\|f \circ g\|_{C^{1,\alpha}} \leq C(\|f\|_{C^{1,\alpha}}, \|g\|_{C^{1,\alpha}})$,

   (iii) $\|f\|_{C^{1,\alpha'}} \leq C\|f\|_{C^\alpha}^{\theta}\|f\|_{C^{1,\alpha}}^{1-\theta}$,

   where $\alpha' \in (0, \alpha)$ and $\theta = \alpha - \alpha'$.