Summer term 2023
Introduction to Fluid Mechanics
Exercise sheet 4
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Exercise 8

Let $\alpha \in (0,1)$.

1. Let $N \in \mathbb{N}$ and $u \in L^\infty_{\text{loc}}(\mathbb{R}; C^{1,\alpha}(\mathbb{R}^N))$. For $t, t' \in \mathbb{R}$ and $y \in \mathbb{R}^N$ let $X(t,t',y)$ be the solution of
$$X(t,t',y) = y + \int_{t'}^{t} u(t'', X(t'',t',y))dt''.$$
   (i) Show that $X_t(y) := X(t,0,y)$ is the Lagrangian trajectory associated to u, and that its inverse is given by $X_t^{-1}(y) = X(0,t,y)$.
   (ii) Show that $X_t^{\pm} - \text{Id} \in C(\mathbb{R}; C^{1,\alpha})$ with
$$\|\nabla X_t^{\pm}\|_{L^\infty} \leq \exp(\int_0^t \|\nabla u\|_{L^\infty} dt'), \quad \|X_t^{\pm} - \text{Id}\|_{C^{1,\alpha}} \leq \exp(C \int_0^t \|u\|_{C^{1,\alpha}} dt').$$
   for all $t \in \mathbb{R}$, and some constant $C > 0$.

2. In the rest of this exercise let $N = 3$. Recall that the solution u_{n+1} of the iteration scheme for the modified Euler equation is given by the formula
$$u_{n+1}(t,x) = u_0(X_{n,t}^{-1}(x)) - \int_0^t A_n(t', X_{n,t'}(X_{n,t}^{-1}(x)))dt', \quad t \in \mathbb{R},\, x \in \mathbb{R}^3,$$
for $n \in \mathbb{N}_0$, where $A_n := A(u_n, u_n)$. Show that there exists a constant $C > 0$ such that for all $t > 0$,
$$\|u_{n+1}\|_{L^\infty([0,t];C^{1,\alpha})} \leq e^{C\int_0^t \|u_n(t')\|_{C^{1,\alpha}}} \|u_0\|_{C^{1,\alpha}} + \int_0^t \|A_n(t',x)\|_{C^{1,\alpha}} e^{C\int_{t'}^t \|u_n\|_{C^{1,\alpha}}} dt'.$$

3. Let $t > 0$ such that $2Ct\|u_0\|_{C^{1,\alpha}} < 1$. Show that
$$\|u_n(t)\|_{C^{1,\alpha}} \leq \frac{\|u_0\|_{C^{1,\alpha}}}{1 - 2C|t|\|u_0\|_{C^{1,\alpha}}} =: C_0$$
for all $n \in \mathbb{N}$.

4. For $n, m \in \mathbb{N}_0$ let $U_{n,m} := u_{n+m} - u_n$. Recall that $U_{n+1,m}$ satisfies
$$\|U_{n+1,m}\|_{L^\infty([0,t];C^\alpha)} \leq \exp(C \int_0^t \|u_{n+m}(t')\|_{C^{1,\alpha}} dt') \cdot \int_0^t \|U_{n,m}(t')\|_{C^\alpha} \|(u_{n+1}, u_{n+m}, u_n)(t')\|_{C^{1,\alpha}} dt'.$$
Deduce that for sufficiently small $T > 0$,
$$\|U_{n,m}\|_{L^\infty([0,T];C^\alpha)} \leq \frac{1}{n!}(1 - 2CT\|u_0\|_{C^{1,\alpha}})^{-n} \|U_{0,m}\|_{L^\infty([0,T];C^\alpha)}, \quad \forall n \in \mathbb{N}_0.$$

5. It follows from the above estimate that there exists a limit $u \in C([0,T]; C^\alpha)$ such that
$$\|u_n - u\|_{L^\infty([0,T];C^{1,\alpha'})} \to 0, \quad \text{as } n \to \infty,$$
for all $\alpha' \in (0,\alpha)$. Show that u solves the modified Euler equation.