Summary term 2023

Karlsruhe Institute of Technology

Introduction to Fluid Mechanics

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Exercise sheet 5

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Exercise 9

We consider the two-dimensional Euler equation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{1}$$

Given a solution $u : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ we define the vorticity $\omega := \partial_{x_1} u^2 - \partial_{x_2} u^1$.

1. Show that ω satisfies the transport equation
$$\partial_t \omega + u \cdot \nabla \omega = 0. \tag{2}$$

2. Show that if u and ω are smooth and decaying sufficiently fast at infinity, then
$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy, \quad x \in \mathbb{R}^2, \tag{3}$$
for fixed $t \in [0,\infty)$, where $x^\perp := \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$.

3. Show that under the same assumptions on u and ω as above, one has
$$\nabla u(x) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{\sigma(x-y)}{|x-y|^2} \omega(y) dy + \frac{1}{2} \omega(x) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad x \in \mathbb{R}^2,$$
where
$$\sigma(z) = \frac{1}{|z|^2} \begin{pmatrix} 2z_1 z_2 & z_2^2 - z_1^2 \\ z_2^2 - z_1^2 & -2z_1 z_2 \end{pmatrix}, \quad z \in \mathbb{R}^2.$$

4. Let $\omega : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a smooth and fast decaying solution of (2). Show that u defined by (3) solves the Euler equation (1) with
$$\nabla \Pi(x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} tr(\nabla u)^2 dy, \quad x \in \mathbb{R}^2.$$

Exercise 10

1. Let v be a weak solution of
$$\begin{cases} \partial_t v + \partial_x (f(v)) = 0, \quad \text{with } f(v) = \frac{1}{2} v^2, \\ v|_{t=0} = v_0, \end{cases} \tag{4}$$
such that $v \in C^1$ on both sides of a C^1-curve $\{x = x(\tau), t = t(\tau) \,|\, \tau \in [a,b]\}$ on the (x,t)-plane. Show that the slope $c(\tau) := \frac{x'(\tau)}{t'(\tau)}$ of this curve satisfies the Rankine-Hugoniot condition
$$c(\tau) = \frac{f(v_+(t(\tau), x(\tau))) - f(v_-(t(\tau), x(\tau)))}{v_+(t(\tau), x(\tau)) - v_-(t(\tau), x(\tau))} = \frac{1}{2}(v_+(t(\tau), x(\tau)) + v_-(t(\tau), x(\tau))).$$

2. We consider equation (4) with the following initial data v_0:

(i) Let $v_0(y) = \begin{cases} 0 & \text{if } y \leq 0, \\ 1 & \text{if } y > 0, \end{cases}$ for $y \in \mathbb{R}$. Show that there exists a continuous solution v of (4) given by

$$v(t,x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x/t & \text{if } 0 \leq x \leq t, \\ 1 & \text{if } x > t. \end{cases}$$

(ii) Let $v_0(y) = \begin{cases} 1 & \text{if } y \leq 0, \\ 0 & \text{if } y > 0, \end{cases}$ for $y \in \mathbb{R}$. Show that there exists a discontinuous weak solution v of (4) given by

$$v(t,x) = \begin{cases} 1 & \text{if } x \leq \frac{1}{2}t, \\ 0 & \text{if } x > \frac{1}{2}t. \end{cases}$$