Summer term 2023
Introduction to Fluid Mechanics
Exercise sheet 6
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Karlsruhe Institute of Technology
JProf. Dr. Xian Liao
M.Sc. Rebekka Zimmermann

Exercise 11

Let $N \in \{2,3\}$, and let $u_0 : \mathbb{R}^N \to \mathbb{R}^N$ be a divergence-free vector field. We consider the classical incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1}$$

Let $u : [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy the modified Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u = Q(u,u), \\ u|_{t=0} = u_0, \end{cases}$$

where $Q(u,u) := -P(u \cdot \nabla u)$, and P denotes the Leray projection operator on the divergence-free vector fields. Show that u is a solution of (1).

Exercise 12

Let $N \in \mathbb{N}$, and let $u_0 \in L^2(\mathbb{R}^N)$. We consider the heat equation

$$\partial_t u - \Delta u = 0, \quad u|_{t=0} = u_0.$$

1. Show that there exists a unique solution, which is given by

$$u(t,x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|\cdot|^2}{4t}} * u_0 = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy,$$

   for $(t,x) \in [0,\infty) \times \mathbb{R}^N$.

2. Let u be given as above. We define the quantities

$$V_m(t) = \sup_{|\alpha|=m} \|D_x^\alpha u(t,x)\|_{L_x^\infty(\mathbb{R}^N)},$$
$$W_m(t) = \sup_{|\alpha|=m} \|D_x^\alpha u(t,x)\|_{L_x^2(\mathbb{R}^N)},$$

   for $m \in \mathbb{N}_0$ and $t \in [0,\infty)$. Show that for all $m \in \mathbb{N}_0$ and $t \in (0,\infty)$ there holds

$$V_m(t) \leq C_m W_0(0) t^{-\frac{2m+N}{4}},$$
$$W_m(t) \leq C_m W_0(0) t^{-\frac{m}{2}},$$

   where $C_m > 0$ are constants depending only on m.

Exercise 13

Let $N \in \{2, 3\}$, and let $u \in L^2_{\text{loc}}([0,\infty) \times \mathbb{R}^N; \mathbb{R}^N)$ be a sufficiently regular solution of the Navier-Stokes equations (1). Show that u is a weak solution, i.e., there holds

$$\begin{aligned}\int_{\mathbb{R}^N} u(t,x) \cdot \varphi(t,x) dx = &\int_0^t \int_{\mathbb{R}^N} (u \cdot \partial_t \varphi + u \otimes u : \nabla \varphi + u \cdot \Delta \varphi) dx dt \\ &+ \int_{\mathbb{R}^N} u_0(x) \cdot \varphi(0,x) dx,\end{aligned}$$

for any $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^N; \mathbb{R}^N)$ with $\operatorname{div} \varphi = 0$.