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Exercise 14

Recall the classical incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1}$$

We consider the regularized Cauchy problem

$$\partial_t v = \Delta v + \mathcal{P}_n Q(v, v), \quad n \in \mathbb{N},$$

where $\mathcal{P}_n = 1_{B_n}(D)$ is the low-frequency cut-off operator, and $Q(v,v) = -P(v \cdot \nabla v)$, with the Leray projection operator P on the divergence-free vector fields. Show that for every $n \in \mathbb{N}$ there exists a unique solution $u_n \in C^1([0,\infty); H_n^{N+1})$, where $H_n^{N+1} = \{v \in H^{N+1} : \operatorname{Supp}(\hat{v}) \subset B_n\}$.

Exercise 15

Let u, v be two weak solutions of (1) with the same initial data u_0, such that both satisfy the energy inequality

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^N)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^N)} dt' \leq \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^N)}, \quad t \in [0, \infty).$$

Then their difference $w = u - v$ satisfies

$$\partial_t w - \Delta w = -P \operatorname{div}(w \otimes u + v \otimes w)$$

in $L^2([0,\infty); H^{-1})$. Show that $w(t, \cdot) = 0$ almost everywhere for all $t \in [0, \infty)$.

Exercise 16

Let $u_0 \in (L^3(\mathbb{R}^3))^3$. Recall that the solution of the heat equation

$$\partial_t u - \Delta u = 0, \quad u|_{t=0} = u_0$$

in dimension $N = 3$ is given by

$$u(t,x) = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|\cdot|^2}{4t}} * u_0(x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^3.$$

Show that for $\beta \geq 3$ and $t > 0$ there holds

$$\|u(t,\cdot)\|_{L^\beta(\mathbb{R}^3)} \leq C t^{-\frac{1}{2}(1-\frac{3}{\beta})} \|u_0\|_{L^3(\mathbb{R}^3)}.$$