Faculty of Mathematics
Institute for Analysis
JProf. Dr. Xian Liao
Office: 3.027
Englerstrasse 2, Geb. 20.30
76131 Karlsruhe
E-Mail: xian.liao@kit.edu
Website: www.math.kit.edu/iana2/~liao
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Lecture notes for "Introduction to Fluid Mechanics" (Lecture 0165900, Summer Semester 2023) [1]

Instructor:
JProf. Xian Liao (xian.liao@kit.edu)

Teaching assistant:
Ms. Rebekka Zimmermann (rebekka.zimmermann@kit.edu)

Time & Place (weekly hours 3+1):

- Monday 09:45-11:15, SR 3.061 (lecture, weekly)
- Friday 14:00-15:30, SR 2.066 (lecture/problem class, each biweekly)

Exam:
Oral exam (02.08.2023-03.08.2023)

[1] Comments are welcome to be sent to me by email.

Contents

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1 Introduction

In this chapter we will introduce the mathematical models which describe the motion of the fluids. In the derivation we will always assume the smoothness of the quantities and domains, unless otherwise clarified. The main reference of this chapter is [4].

1.1 Derivation of mathematical models

The description of an evolutionary fluid (liquid or gas) involves $(N + 2)$ evolution equations for $(N + 2)$ fields, namely

the mass density $\rho \geqslant 0$, the velocity field $u \in \mathbb{R}^N$ and the energy $e \geqslant 0$.

We recall the standard derivation of the evolution equations in eulerian form in the case of a fluid filling the whole space $\mathbb{R}^N$. They follow from the principles of conservation of mass, momentum and energy.

1.1.1 Conservation of mass and momentum

Let $t \geqslant 0$, $x \in \mathbb{R}^N$ denote the time and space variables. Let $\Omega \subset \mathbb{R}^N$ be arbitrary smooth volume.

Conservation of mass. By conservation of mass, the variation of mass inside Ω:

$$\frac{d}{dt}\int_\Omega \rho(t,x)dx = \int_\Omega \partial_t \rho(t,x)dx$$

is equal to the flux of mass on $\partial\Omega$:

$$-\int_{\partial\Omega} \rho(t,x)\, u(t,x) \cdot n\, d\sigma$$

where n denotes the unit outer normal to $\partial\Omega$. By Gauss' Theorem (or Stokes' Formular),

$$-\int_{\partial\Omega} \rho(t,x)\, u(t,x) \cdot n\, d\sigma = -\int_\Omega \operatorname{div}\left(\rho(t,x)\, u(t,x)\right) dx$$

where $\operatorname{div} F := \sum_{j=1}^N \partial_{x_j} F^j$ for $F(x) = \begin{pmatrix} F^1(x) \\ \vdots \\ F^N(x) \end{pmatrix}$, we deduce that

$$\int_\Omega \Big(\partial_t \rho(t,x) + \operatorname{div}\,(\rho(t,x)u(t,x))\Big) dx = 0.$$

Since Ω is arbitrary, we have derived the continuity equation for the density function:

$$\partial_t \rho + \operatorname{div}\,(\rho u) = 0. \tag{1.1}$$

Conservation of momentum. Similarly as above, the conservation of momentum ρu, $u = \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix}$ implies

$$\frac{d}{dt}\int_\Omega (\rho u^j) dx = -\int_{\partial\Omega} (\rho u^j)(u\cdot n) d\sigma + \int_\Omega \rho f^j dx + \int_{\partial\Omega} (\Sigma \cdot n)^j d\sigma, \quad j = 1, \cdots, N.$$

Here $f \in \mathbb{R}^N$ denotes the possible external forces acting on the fluid, e.g. gravity, Coriolis, electromagnetic forces, surface forces [2]. The tensor $\Sigma \in \mathbb{R}^{N\times N}$ is called Cauchy stress tensor, and two common stresses in a fluid are caused by compression and viscous effects respectively (Stokes law):

$$\Sigma = -p\mathrm{Id}_{N\times N} + \tau,$$

where $p \in \mathbb{R}$ is the pressure and $\tau \in \mathbb{R}^{N\times N}$ is the *symmetric* viscous stress tensor:

$$\tau = \tau(Du, \rho, \theta),$$

where θ denotes the temperature. If we assume that τ is a linear function of Du, invariant under translation/rotation and that the fluid is isotropic (i.e. we consider *newtonian* fluids [3]), then

$$\tau = \lambda(\operatorname{div} u)\mathrm{Id} + 2\mu d = \lambda(\operatorname{div} u)\mathrm{Id} + \mu(\nabla u + (\nabla u)^T), \quad d := \frac{1}{2}(\nabla u + (\nabla u)^T),$$

where λ, μ denote the Lamé viscosity coefficients:

$$\lambda = \lambda(\rho, \theta), \quad \mu = \mu(\rho, \theta),$$

[2] They may occur due to the fluid particles lying outside Ω.

[3] There exist non-newtonian fluids in our life, and common examples could be ketchup, toothpaste, blood, etc.

and satisfy

$$\mu \geqslant 0, \quad \lambda + \frac{2}{N}\mu \geqslant 0. \tag{1.2}$$

One can rewrite

$$\tau = K(\operatorname{div} u)\mathrm{Id} \ + 2\mu\Big(\frac{1}{2}(\nabla u + (\nabla u)^T) - \frac{1}{N}\operatorname{div} u\mathrm{Id}\Big), \quad K := \lambda + \frac{2}{N}\mu,$$

where the first summand corresponds to the compression effect and the second trace-free tensor corresponds to deformation/shear effect. The parameter μ is referred to as the dynamic/kinetic viscosity (coefficient) or the first viscosity or simply the viscosity, while K is referred to as the bulk/volume viscosity or the second viscosity.
If $\lambda = \mu = 0$ such that $\tau = 0$, we are in the inviscid case, while if $\mu > 0$ and $\lambda + \mu > 0$, the fluid is viscous.
Hence we arrive at the evolution equation for the momentum ρu:

$$\partial_t(\rho u^j) + \sum_{k=1}^{N} \partial_{x_k}\big(\rho u^j u^k - \tau_{jk}) + \partial_{x_j} p = \rho f^j, \quad j = 1, \cdots, N, \tag{1.3}$$

or in a compact form

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\tau + \nabla p = \rho f. \tag{1.4}$$

By use of the continuity equation (1.1), we can rewrite the above equation in the following form

$$\rho\partial_t u + \rho u \cdot \nabla u + \operatorname{div}\tau + \nabla p = \rho f. \tag{1.5}$$

1.1.2 Energy equations

We assume that the fluctuations around thermodynamic equilibria are sufficiently weak so that the thermodynamical state of the fluid is determined by the state variables as in classical thermodynamics:

thermodynamic pressure p, internal energy per unit mass e, thermodynamic temperature θ, mass density ρ

Conservation of energy: First law of thermodynamics. As the total energy E consists of the kinetic energy $\rho|u|^2/2$ and the internal energy ρe:

$$E = \frac{1}{2}\rho|u|^2 + \rho e,$$

the conservation of energy (i.e. first law of thermodynamics) yields

$$\frac{d}{dt}\int_\Omega \rho(\frac{1}{2}|u|^2+e)dx = -\int_{\partial\Omega}\rho(\frac{1}{2}|u|^2+e)(u\cdot n)d\sigma$$
$$+\int_\Omega \rho f\cdot u dx + \int_{\partial\Omega} u\cdot(\Sigma\cdot n)d\sigma - \int_{\partial\Omega} q\cdot n d\sigma,$$

where the second and third integrals on the righthand side denote the work done by the forces, and the last integral denotes the heat transferred by the heat flux q. By Gauss' Theorem one arrives at the evolution equation **(Exercise)**

$$\partial_t\big(\rho(\frac{1}{2}|u|^2+e)\big)+\mathrm{div}\,\Big(u\Big[\rho(\frac{1}{2}|u|^2+e)+p\Big]\Big) = \mathrm{div}\,(\tau\cdot u)-\mathrm{div}\,(q)+\rho f\cdot u, \quad (1.6)$$

and furthermore, by view of the continuity equation (1.1) and the momentum equation (1.4), we derive

$$\partial_t(\rho e)+\mathrm{div}\,(\rho u e)+p\mathrm{div}\,u = -\mathrm{div}\,(q)+\tau : d \quad (1.7)$$

or

$$\rho\partial_t e+\rho u\cdot\nabla e+p\mathrm{div}\,u = -\mathrm{div}\,(q)+\tau : d. \quad (1.8)$$

where $A:B=\sum_{j,k=1}^N A_{jk}B_{jk}$ for two matrices $A=(A_{jk})_{1\leqslant j,k\leqslant N}$ and $B=(B_{jk})_{1\leqslant j,k\leqslant N}$.

State equations & Navier-Stokes-Fourier equations. To close the system, we have to postulate the relations among ρ,θ,p,e,q. Let ρ,θ be two independent thermodynamic state variables.

Let

$$p=p(\rho,\theta),\quad e=e(\rho,\theta) \quad (1.9)$$

be given by general constitutive laws. Let

$$q=-\kappa(\rho,\theta,|\nabla\theta|)\nabla\theta \quad (1.10)$$

be the heat flux given by Fourier law. The heat conduction coefficient κ may depend on $\rho,\theta,|\nabla\theta|$, and in most cases depends only on ρ,θ, or even is taken to be constant.

To conclude, we have the following $(N+2)$-evolution equations (1.1)-(1.4)-(1.7) for $(N+2)$-variables (ρ,u,e):

$$\begin{cases} \partial_t\rho+\mathrm{div}\,(\rho u)=0, \\ \partial_t(\rho u^j)+\mathrm{div}\,(\rho u^j u)-\sum_{k=1}^N\partial_{x_k}(\mu(\partial_{x_k}u^j+\partial_{x_j}u^k)) \\ \qquad\qquad -\partial_{x_j}(\lambda\mathrm{div}\,u)+\partial_{x_j}p=\rho f^j,\quad j=1,\cdots,N, \\ \partial_t(\rho e)+\mathrm{div}\,(\rho u e)+p\mathrm{div}\,u=\mathrm{div}\,(\kappa\nabla\theta)+2\mu d:d+\lambda(\mathrm{div}\,u)^2, \end{cases} \quad (1.11)$$

where p, e are given in terms of ρ, θ in the state equations (1.9), and the viscosity and heat conduction coefficients μ, λ, κ may depend on ρ, θ. It is called (compressible) Navier-Stokes(-Fourier) equations.

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Second law of Thermodynamics. We postulate the existence of a new state variable: the specific entropy

$$s = s(\rho, \theta), \tag{1.12}$$

which satisfies

$$\frac{\partial s}{\partial \theta} = \frac{1}{\theta}\frac{\partial e}{\partial \theta}, \quad \frac{\partial s}{\partial \rho} = \frac{1}{\theta}\Big(\frac{\partial e}{\partial \rho} - \frac{p}{\rho^2}\Big).$$

Then we have the entropy equation from (1.1) and (1.7) **(Exercise)**:

$$\partial_t(\rho s) + \mathrm{div}\,\Big(\rho u s + \frac{q}{\theta}\Big) = \frac{1}{\theta}\tau : d - \frac{1}{\theta^2} q \cdot \nabla\theta. \tag{1.13}$$

By virtue of the second law of thermodynamics, the righthand side should be nonnegative. Notice that by the decomposition of a matrix into a multiple identity matrix and a trace-free matrix

$$\begin{aligned}\tau : d &= \Big((\lambda + \frac{2}{N}\mu)\mathrm{div}\, u\mathrm{Id} + 2\mu(d - \frac{1}{N}\mathrm{div}\, u\mathrm{Id})\Big) : \Big(\frac{1}{N}\mathrm{div}\, u\mathrm{Id} + (d - \frac{1}{N}\mathrm{div}\, u\mathrm{Id})\Big) \\ &= (\lambda + \frac{2}{N}\mu)\frac{1}{N}(\mathrm{div}\, u)^2 + 2\mu(d - \frac{1}{N}\mathrm{div}\, u\mathrm{Id}) : (d - \frac{1}{N}\mathrm{div}\, u\mathrm{Id}).\end{aligned}$$

This gives the restriction:

$$\mu \geqslant 0, \quad K = \lambda + \frac{2}{N}\mu \geqslant 0, \quad q \cdot \nabla\theta \leqslant 0,$$

that is, (1.2) and $\kappa \geqslant 0$ in (1.10). For common fluids (which e.g. do not move too fast), experiments show that $K = \lambda + \frac{2}{N}\mu$ is very small and could be taken as zero in the simulation. Nevertheless in the study of sound waves or shock waves which transport in fast-moving compressible fluids it plays an important role.

1.2 Simplified models

1.2.1 Barotropic models.

In the case of ideal gas, the constitutive equations (1.9) read

$$p = (\gamma - 1)\rho e, \quad e = C_v \theta, \tag{1.14}$$

where $\gamma > 1$ is the adiabatic constant, and $C_v > 0$ is the thermo capacity at constant volume. Often one denotes by $R = C_v(\gamma - 1)$ the ideal gas constant, and by $C_p = \gamma C_v$ the thermo capacity at constant pressure.

Example 1.1. • *Isentropic compressible fluids. In the case of ideal gas, the entropy* (1.12) *takes the form (up to a constant)*

$$s = C_v\big(\log(e) + (1-\gamma)\log(\rho)\big).$$

If $s = s_0 = const.$ all the time and consequently,

$$p(\rho) = a\rho^\gamma, \quad a = (\gamma - 1)\exp(s_0/C_v) > 0,$$

then (1.1)-(1.4) *represent a closed system for $(N+1)-$variables (ρ, u) describing the motion of an isentropic compressible viscous fluid:*

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u^j) + \mathrm{div}\,\big(\rho u^j u\big) - \sum_{k=1}^N \partial_{x_k}(\mu(\partial_{x_k} u^j + \partial_{x_j} u^k)) \\ \qquad -\partial_{x_j}(\lambda \mathrm{div}\, u) + \partial_{x_j}(a\rho^\gamma) = \rho f^j, \quad j = 1, \cdots, N. \end{cases} \tag{1.15}$$

The total energy of the flow reads

$$E = \frac{1}{2}\rho|u|^2 + P(\rho),$$

where

$$P'(z)z - P(z) = p(z).$$

The energy equation (1.6) *is then a consequence of* (1.1)-(1.4) *if $\lambda = \mu = \kappa = 0$* ***(Exercise)****. Thus it suffices to solve the system* (1.15)*. The system* (1.15) *is sometimes simply called compressible Navier-Stokes equations.*

It is possible to deduce from the kinetic theory of gases that $\gamma = \frac{N+2}{N}$, e.g. $\gamma = \frac{5}{3}$ if $N = 3$, for a monatomic gas. The physical relevant case is $\gamma \in (1, \frac{5}{3}]$.

- *Isothermal compressible fluids. Similarly, if we suppose $\theta(t) = \theta_0 = const.$, then* (1.14) *implies*

$$p = R\rho\theta_0.$$

Then (1.15) *with $a\rho^\gamma$ replaced by $R\rho\theta_0$ describes the motion of isenthermal compressible fluids.*

- *Barotropic flows: the pressure p depends solely on the density ρ:*

$$p = p(\rho),$$

and the fluid motion is described by (1.15) *with $a\rho^\gamma$ replaced by $p(\rho)$. The isentropic/isothermal ideal gases are special examples.*

1.2.2 Incompressible models

Lagrangian viewpoint We have derived the evolution equations for the fluid motion in eulerian form, where one fixes a point $x \in \mathbb{R}^N$ and observe the fluid flows as time evolves (Eulerian viewpoint). Nevertheless one can follow directly a specific fluid parcel $y \in \mathbb{R}^N$ (Lagrangian viewpoint).
Let $X(t, y)$ be the integral curves[4]

$$\begin{cases} \partial_t X(t,y) = u(t, X(t,y)), \\ X(t,y)|_{t=0} = y, \end{cases} \tag{1.16}$$

and we call $X_t = X(t, \cdot)$ the Lagrangian trajectory. Let $J(t, y) = \det(\nabla_y X_t)$ be the jacobian of the transformation ($y \mapsto X_t(y) = X(t, y)$), such that **(Exercise)**

$$\begin{cases} \partial_t J(t,y) = \operatorname{div} u(t, X(t,y)) J(t,y), \\ J(t,y)|_{t=0} = 1, \end{cases} \tag{1.17}$$

and hence

$$J(t, y) = 1 + t \operatorname{div} u(0, y) + o(|t|), \quad \text{ as } |t| \to 0.$$

Let the initial time be any fixed time t, $X(t + h, y)$ be the integral curve

$$\begin{cases} \partial_h X(t+h,y) = u(t+h, X(t+h,y)), \\ X(t,y) = y, \end{cases} \tag{1.18}$$

and $J(t+h, y)$ be the jacobian of the transformation ($y \mapsto X(t+h, y)$). Then the conservation of mass at time t and $t + h$:

$$\begin{aligned} \int_{\Omega(t)} \rho(t,y) dy &= \int_{\Omega(t+h)} \rho(t+h, x) dx, \quad \Omega(t+h) := \{X(t+h,y) \,|\, y \in \Omega(t)\} \\ &= \int_{\Omega(t)} \rho(t+h, X(t+h,y)) J(t+h,y) dy \end{aligned}$$

[4]If the velocity vector field $u : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is smooth enough, e.g.

$$\begin{aligned} &u \in L^1_{\text{loc}}(\mathbb{R}; \text{Lip}\,(\mathbb{R}^N, \mathbb{R}^N)), \\ &\text{i.e. } \| \|\nabla_x u\|_{L^\infty_x(\mathbb{R}^N;\mathbb{R}^{N\times N})} \|_{L^1_t(I)} < \infty, \quad \forall \text{ finite interval } I \subset \mathbb{R}, \end{aligned}$$

then the Cauchy-Lipschitz theorem implies the unique flow

$$X_t(\cdot) = X(t, \cdot) : \mathbb{R}^N \mapsto \mathbb{R}^N,$$

which is defined as the solution of initial value problem of the ordinary differential equation (1.16) (with $y \in \mathbb{R}^N$ viewed as a parameter) implies (noticing $\Omega(t)$ is arbitrary)

$$\rho(t,y) = \rho(t+h, X(t+h,y))J(t+h,y).$$

Therefore as $h \to 0_+$ we obtain

$$\rho(t,y) = \rho(t,y) + h\Big(\partial_t\rho(t,y) + u(t,y)\cdot\nabla\rho(t,y) + \rho(t,y)\mathrm{div}\, u(t,y)\Big) + o(h),$$

and hence the coefficient of h should vanish:

$$\partial_t\rho + u\cdot\nabla\rho + \rho\mathrm{div}\, u = 0,$$

which is exactly the continuity equation (1.1).

Incompressibility condition and incompressible models. Many common liquids are incompressible (or only very slightly compressible), that is, the volume of an open set $\Omega(t)$ at some fixed time t should be the same as the volume of the transported set

$$\Omega(t+h) := \{X(t+h,y) \,|\, y \in \Omega(t)\},$$

which reads more precisely as

$$\int_{\Omega(t)} dy = \int_{\Omega(t+h)} dx = \int_{\Omega(t)} J(t+h,y)dy, \quad \forall t, h. \tag{1.19}$$

That is,

$$1 = \det(\nabla_y X(t,y)) = J(t,y), \quad \forall t, y. \tag{1.20}$$

or equivalently,

$$\mathrm{div}\, u(t,x) = 0, \quad \forall t, x. \tag{1.21}$$

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If $\mathrm{div}\, u = 0$, then the pressure Π [5] is in fact a Lagrangian multiplier associated to (1.21)[6]. The equations (1.1)-(1.4) together with the (1.21) represent a

[5] The pressure Π here is not necessarily the thermodynamic pressure. Notice that only $\nabla\Pi$ (instead of Π itself) appears in the momentum equation, and the system does not change if one modifies the pressure by a constant. In particular, it can not be recovered simply by applying the constitutive laws for fluids.

In the zero Mach number limit $\varepsilon \to 0$, one can expand the thermodynamic pressure $p = p_0 + \varepsilon^2\Pi + o(\varepsilon^2)$ where p_0 is a constant. Then one recovers the incompressible model (1.22) from the compressible model (1.11).

[6] Notice that $\sigma_{ij}\partial_i u_j = 0$ for all u such that $\mathrm{div}\, u = 0$ if and only if $\sigma_{ij} = \Pi\delta_{ij}$ for some Π.

closed system for $(N+2)-$variables (ρ, u, Π) describing the motion of an incompressible viscous fluid:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u^j) + \operatorname{div}\left(\rho u^j u\right) - \sum_{k=1}^N \partial_{x_k}(\mu(\partial_{x_k} u^j + \partial_{x_j} u^k)) + \partial_{x_j}\Pi = \rho f^j, \\ \qquad\qquad j = 1, \cdots, N, \\ \operatorname{div} u = 0. \end{cases} \tag{1.22}$$

Observe that for incompressible fluids, if the solutions are smooth enough, e.g. $u(t,x) \in L^1_{\text{loc}}(\mathbb{R}_+; \text{Lip}(\mathbb{R}^N))$, the continuity equation reduces to

$$\partial_t \rho + u \cdot \nabla \rho = 0,$$

which admits a unique solution [7] **(Exercise)**

$$\rho(t, X(t,y)) = \rho_0(y), \quad \text{i.e. } \rho(t,x) = \rho_0(X_t^{-1}(x)).$$

If initially $\rho_0 = 1$ is a constant, then $\rho(t,x) = 1$ for all the times, and we call it a homogeneous fluid. If the density ρ is not a constant, then (1.22) are called inhomogeneous (or density-dependent) incompressible Navier-Stokes equations.

Incompressible homogeneous models. In the homogeneous case $\rho = 1$, the mass conservation law $\partial_t \rho + \operatorname{div}(\rho u) = 0$ is equivalent to the incompressibility condition $\operatorname{div} u = 0$.

The viscosity coefficient μ is then a constant. If $\mu > 0$, then (1.22) becomes the (classical) incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla \Pi = f, \\ \operatorname{div} u = 0, \end{cases} \tag{1.23}$$

which describes the motion of the homogeneous incompressible viscous fluids. If $\mu = 0$, then (1.22) becomes the (classical) incompressible Euler equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = f, \\ \operatorname{div} u = 0, \end{cases} \tag{1.24}$$

which describes the motion of the homogeneous incompressible inviscid fluids. If (ρ, u) are known, then the energy equation (1.7) becomes

$$\partial_t(\rho e) + \operatorname{div}(\rho u e) - \operatorname{div}(\kappa \nabla \theta) = \frac{1}{2}\mu(\partial_i u^j + \partial_j u^i)^2,$$

[7] Indeed it is just the Lagrangian formulation of the above transport equation with divergence-free velocity field.

and in particular in the homogeneous case $\rho = 1$, $e = e(1,\theta) = C_v\theta$, the above equation becomes the transport-diffusion equation for the temperature θ.
Incompressible (inhomogeneous) perfect fluids. The evolution of incompressible perfect fluids is described by the following equations (i.e. (1.22) with $\mu = 0$ and $f^j = -\partial_{x_j} F$)

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho\big(\partial_t u + u \cdot \nabla u + \nabla F\big) + \nabla \Pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{1.25}$$

If we consider the perfect fluids in some bounded smooth domain Ω and assume the impermeability condition on the boundary:

$$(u \cdot n)|_{\partial\Omega} = 0, \tag{1.26}$$

where n denotes the outer normal vector on $\partial\Omega$, then the incompressibility condition means that, for each time t, $X(t,\cdot)$ is a smooth diffeomorphism from Ω to itself that preserves the orientation and volume (recalling (1.20)). By use of Lagrangian coordinates, the system (1.25) reduces to (recalling $\rho(t, X(t,y))) = \rho_0(y))$

$$\begin{cases} \rho_0(y)\Big(\partial_t^2 X(t,y) + \nabla_x F(t, X(t,y)\Big) + \nabla_x \Pi(t, X(t,y)) = 0, \\ X(0,y) = y, \quad \partial_t X(0,y) = u_0(y), \\ X(t,\cdot) \in \{\gamma : \Omega \to \Omega \text{ diffeomorphism s.t. } \det(\nabla\gamma) = 1\}, \end{cases} \tag{1.27}$$

where (ρ_0, u_0) are the initial data at time 0. This is related to Least Action Principle (a variational problem, formulated by V.I.Arnold 1960s): The Action is the sum of the kinetic energy and the potential energy

$$A(t,X) = \int_\Omega \rho_0(y)\Big(\frac{1}{2}|\partial_t X(t,y)|^2 - F(t, X(t,y))\Big)dy,$$

and the Least Action Principle says that if $t_1 - t_0 > 0$ is not too large, then

$$\int_{t_0}^{t_1} A(t,X)dt \leqslant \int_{t_0}^{t_1} A(t,\gamma)dt$$

holds for all flow map $\gamma(t,\cdot)$, which is an orientation and volume-preserving diffeomorphism such that $\gamma(t_0) = X(t_0)$, $\gamma(t_1) = X(t_1)$, i.e. the Action integrand from t_0 to t_1 is minimal for X. The resolution of (1.27) is related to the shortest patch problem.

2 Euler equations

In this chapter we discuss the (classical) incompressible Euler equations for the motion of perfect incompressible fluid flows (without external forces) given in (1.24) mainly in dimension $N = 2$ or 3

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{2.1}$$

2.1 Vorticity

In this section we will discuss the vorticity and we restrict ourselves in three-dimensional case $N = 3$. The main reference is [5].

2.1.1 Vorticity-Transport formula

Notice that the 3×3-matrix $U := \nabla u = (\partial_{x_j} u^i)$ can be decomposed into a symmetric part d (deformation tensor) and an antisymmetric part a (rotation matrix):

$$\begin{aligned} \nabla u = d + a &:= \frac{1}{2}(\nabla u + (\nabla u)^T) + \frac{1}{2}(\nabla u - (\nabla u)^T) \\ &= \begin{pmatrix} \partial_{x_1} u^1 & \frac{1}{2}(\partial_{x_1} u^2 + \partial_{x_2} u^1) & \frac{1}{2}(\partial_{x_1} u^3 + \partial_{x_3} u^1) \\ \frac{1}{2}(\partial_{x_1} u^2 + \partial_{x_2} u^1) & \partial_{x_2} u^2 & \frac{1}{2}(\partial_{x_2} u^3 + \partial_{x_3} u^2) \\ \frac{1}{2}(\partial_{x_1} u^3 + \partial_{x_3} u^1) & \frac{1}{2}(\partial_{x_2} u^3 + \partial_{x_3} u^2) & \partial_{x_3} u^3 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & \frac{1}{2}(\partial_{x_2} u^1 - \partial_{x_1} u^2) & \frac{1}{2}(\partial_{x_3} u^1 - \partial_{x_1} u^3) \\ \frac{1}{2}(\partial_{x_1} u^2 - \partial_{x_2} u^1) & 0 & \frac{1}{2}(\partial_{x_3} u^2 - \partial_{x_2} u^3) \\ \frac{1}{2}(\partial_{x_1} u^3 - \partial_{x_3} u^1) & \frac{1}{2}(\partial_{x_2} u^3 - \partial_{x_3} u^2) & 0 \end{pmatrix} \end{aligned}$$

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We apply ∇ to the u-equation in (2.1) to arrive at the following equation for the matrix $U = \nabla u = (\partial_{x_j} u^i)$

$$\partial_t U + u \cdot \nabla U + U^2 + \nabla^2 \Pi = 0.$$

We have decomposed U into symmetric part $d = \frac{1}{2}(U + U^T)$ and antisymmetric part $a = \frac{1}{2}(U - U^T)$, such that U^2 can be decomposed into symmetric and antisymmetric parts:

$$U^2 = (d^2 + a^2) + (da + ad).$$

The symmetric part for U-equation reads as

$$\partial_t d + u \cdot \nabla d + d^2 + a^2 + \nabla^2 \Pi = 0, \tag{2.2}$$

while the antisymmetric part reads as

$$\partial_t a + u \cdot \nabla a + da + ad = 0. \tag{2.3}$$

The vorticity ω of the velocity field u is given by

$$\omega = \mathrm{curl}\,(u) = \begin{pmatrix} \partial_{x_2} u^3 - \partial_{x_3} u^2 \\ \partial_{x_3} u^1 - \partial_{x_1} u^3 \\ \partial_{x_1} u^2 - \partial_{x_2} u^1 \end{pmatrix}$$

and satisfies

$$ah = \frac{1}{2}\omega \times h, \quad \forall h \in \mathbb{R}^3 .$$

If $\mathrm{div}\, u = 0$, then $tr(d) = 0$, and the equation (2.3) is equivalent to the following equation for the vorticity $\omega \in \mathbb{R}^3$ (**Exercise.**)

$$\partial_t \omega + u \cdot \nabla \omega = d\omega, \tag{2.4}$$

or equivalently,

Lemma 2.1. *Let $N = 3$. If the velocity field u satisfies* (2.1) *together with some pressure term, then its curl $\omega = \mathrm{curl}\,(u)$ satisfies*

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u. \tag{2.5}$$

The righthand side of (2.5) is called the vortex stretching term, which amplifies the vorticity when the velocity is diverging in the direction of ω.
Recall the definition of the trajectory $X(t, y)$ in (1.16)

$$\begin{cases} \partial_t X(t, y) = u(t, X(t, y)), \\ X(t, y)|_{t=0} = y. \end{cases} \tag{2.6}$$

Then the solution to (2.5) is given by (**Exercise**)

$$\omega(t, X(t, y)) = \nabla_y X(t, y)\omega_0(y) = (\omega_0(y) \cdot \nabla_y) X(t, y), \tag{2.7}$$

where ω_0 denotes the initial vorticity. It is however in general open to solve (2.5), and one notices that the definition of the trajectory $X(t, y)$ depends on the velocity $u(t, x)$, which in turn depends on $\omega(t, x)$ (by Biot-Savart's law, see later). We have nevertheless some special solutions of (2.1) below.

2.1.2 Special solutions

Any real, symmetric and trace-free 3×3 matrix will determine a solution to the Euler equations (2.1).

Lemma 2.2. *Let $N = 3$. Let $d = d(t)$ be a real, symmetric and trace-free 3×3 matrix. Let $\omega = \omega(t)$ be determined by the ODE equation on $\mathbb{R}^3$:*

$$\frac{d}{dt}\omega = d\omega, \quad \omega|_{t=0} = \omega_0 \in \mathbb{R}^3 . \tag{2.8}$$

Then

$$(u, \Pi)(t, x) = \Big(\frac{1}{2}\omega \times x + d\, x,\ -\frac{1}{2}\big(\partial_t d + d^2 + a^2\big)x \cdot x\Big) \tag{2.9}$$

is a solution to (2.1). Here the antisymmetric matrix a is defined by $ah = \frac{1}{2}\omega \times h$.

Proof. For $d = d(t)$ and $\omega = \omega(t)$ given by (2.8), we define the velocity as in (2.9):

$$u(t, x) = \frac{1}{2}\omega \times x + d\, x,$$

such that **(Exercise)**

$$\operatorname{div} u = 0, \quad \operatorname{curl} u = \omega, \quad \frac{1}{2}(\nabla u + \nabla^T u) = d, \quad ah := \frac{1}{2}(\nabla u - \nabla^T u)h = \frac{1}{2}\omega \times h,$$

and (2.4) (and hence (2.3)) holds. With the choice of Π in (2.9)

$$\Pi = -\frac{1}{2}\big(\partial_t d + d^2 + a^2\big)x \cdot x,$$

the equation (2.2) holds correspondingly. Thus the pair (2.9) satisfies (2.1). □

Example 2.3. *We give some examples of the exact solutions of (2.1) that illustrate the interactions between a rotation and a deformation.*

1. *Jet flows. Let $\gamma_1, \gamma_2 > 0$ and*

$$\omega_0 = 0 \in \mathbb{R}^3, \quad d = \begin{pmatrix} -\gamma_1 & 0 & 0 \\ 0 & -\gamma_2 & 0 \\ 0 & 0 & (\gamma_1 + \gamma_2) \end{pmatrix}.$$

Then by Lemma 2.2, $\omega(t) = 0$, and the pair

$$(u, \Pi)(t, x) = \left(\begin{pmatrix} -\gamma_1 x_1 \\ -\gamma_2 x_2 \\ (\gamma_1 + \gamma_2) x_3 \end{pmatrix}, -\frac{1}{2}\big(\gamma_1^2 x_1^2 + \gamma_2^2 x_2^2 + (\gamma_1 + \gamma_2)^2 x_3^2\big) \right)$$

is a solution to (2.1).

The flow forms two jets along the positive and negative directions of x_3-axis, along the particle trajectories $X(t, y)$ (recalling (1.16)*)*

$$X(t, y) = \begin{pmatrix} e^{-\gamma_1 t} & 0 & 0 \\ 0 & e^{-\gamma_2 t} & 0 \\ 0 & 0 & e^{(\gamma_1+\gamma_2)t} \end{pmatrix} y = \begin{pmatrix} e^{-\gamma_1 t} y_1 \\ e^{-\gamma_2 t} y_2 \\ e^{(\gamma_1+\gamma_2)t} y_3 \end{pmatrix}.$$

A jet flow is axisymmetric flow without swirl.

2. *Strain flows. Let $\gamma > 0$, and*

$$\omega_0 = 0 \in \mathbb{R}^3, \quad d = \begin{pmatrix} -\gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then by Lemma 2.2, $\omega(t) = 0$, and the pair

$$(u, \Pi)(t, x) = \left(\begin{pmatrix} -\gamma x_1 \\ \gamma x_2 \\ 0 \end{pmatrix}, -\frac{1}{2}\big(\gamma^2 x_1^2 + \gamma^2 x_2^2\big) \right)$$

is a solution to (2.1). *The particle trajectories read*

$$X(t, y) = \begin{pmatrix} e^{-\gamma t} y_1 \\ e^{\gamma t} y_2 \\ y_3 \end{pmatrix}.$$

The strain flow is independent of x_3.

3. *Vortex flows. Let $\alpha \in \mathbb{R}$ and*

$$\omega_0 = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \quad d = 0 \in \mathrm{Id}_{3\times 3}.$$

Then by Lemma 2.2, $\omega(t) = \omega_0$, and the pair

$$(u, \Pi)(t, x) = \left(\begin{pmatrix} -\frac{1}{2}\alpha x_2 \\ \frac{1}{2}\alpha x_1 \\ 0 \end{pmatrix}, \frac{1}{8}\alpha^2 (x_1^2 + x_2^2) \right)$$

is a solution to (2.1)*. The particle trajectories read*

$$X(t,y) = \begin{pmatrix} \cos(\varphi_t) & -\sin(\varphi_t) & 0 \\ \sin(\varphi_t) & \cos(\varphi_t) & 0 \\ 0 & 0 & 1 \end{pmatrix} y = \begin{pmatrix} \cos(\varphi_t)y_1 - \sin(\varphi_t)y_2 \\ \sin(\varphi_t)y_1 + \cos(\varphi_t)y_2 \\ y_3 \end{pmatrix}, \quad \varphi_t = \frac{1}{2}\alpha t.$$

This vortex flow is independent of x_3*-variable, and rotates on the* (x_1, x_2)*-plane.*

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4. *Rotation jets. We take the superposition of a jet and a vortex:*

$$\omega_0 = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \quad d = \begin{pmatrix} -\gamma_1 & 0 & 0 \\ 0 & -\gamma_2 & 0 \\ 0 & 0 & (\gamma_1+\gamma_2) \end{pmatrix}.$$

Then by Lemma 2.2, $\omega(t) = \begin{pmatrix} 0 \\ 0 \\ e^{(\gamma_1+\gamma_2)t}\alpha \end{pmatrix}$*, and the pair*

$$(u,\Pi)(t,x) = \left(\begin{pmatrix} -\gamma_1 x_1 - \frac{1}{2}e^{(\gamma_1+\gamma_2)t}\alpha x_2 \\ -\gamma_2 x_2 + \frac{1}{2}e^{(\gamma_1+\gamma_2)t}\alpha x_1 \\ (\gamma_1+\gamma_2)x_3 \end{pmatrix}, \textbf{\textit{Exercise}} \right)$$

is a solution to (2.1)*. The particle trajectories read*

$$X(t,y) = \begin{pmatrix} X_1(t,y) \\ X_2(t,y) \\ e^{(\gamma_1+\gamma_2)t}y_3 \end{pmatrix},$$

where the first two components satisfy the following ODE:

$$\partial_t \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} -\gamma_1 & -\frac{1}{2}e^{(\gamma_1+\gamma_2)t}\alpha \\ \frac{1}{2}e^{(\gamma_1+\gamma_2)t}\alpha & -\gamma_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

and in particular $\partial_t(X_1^2 + X_2^2) = -2\gamma_1 X_1^2 - 2\gamma_2 X_2^2$*, such that*

$$e^{-2\max(\gamma_1,\gamma_2)t}(y_1^2 + y_2^2) \leqslant (X_1^2 + X_2^2)(t,y) \leqslant e^{-2\min(\gamma_1,\gamma_2)t}(y_1^2 + y_2^2).$$

A rotating jet is axisymmetric flow with swirl.

Example 2.4 (Beltrami flows). *Any steady, divergence-free velocity field* $u(x) \in \mathbb{R}^3$ *that satisfies the Beltrami condition*

$$\omega(x) = \lambda(x)u(x) \text{ for some } \lambda(x) \neq 0 \tag{2.10}$$

is a (steady) solution to (2.1). *Indeed, if some divergence-free velocity* $u(x)$ *and its vorticity* $\omega(x) = \operatorname{curl}(u(x))$ *satisfy* (2.10), *then*

$$0 = \operatorname{div} \omega = u \cdot \nabla \lambda + \lambda \operatorname{div} u = u \cdot \nabla \lambda.$$

Hence the (steady) vorticity equation (2.5) *is satisfied:*

$$u \cdot \nabla \omega = u \cdot \nabla(\lambda u) = (u \cdot \nabla \lambda)u + \lambda u \cdot \nabla u = 0 + \omega \cdot \nabla u.$$

Therefore, by Corollary 2.12, $u(x)$ *and the associated* $\nabla \Pi$ *solves* (2.1).
One typical example is the celebrated Arnold-Beltrami-Childress periodic flow

$$u(x) = \begin{pmatrix} A\sin(x_3) + C\cos(x_2) \\ B\sin(x_1) + A\cos(x_3) \\ C\sin(x_2) + B\cos(x_1) \end{pmatrix}.$$

2.2 A dip on analysis and Biot-Savart's law

In this section we recall some definitions and facts from analysis lectures[8], which will help to understand Biot-Savart's law: A formula for the divergence-free velocity field in terms of its vorticity.

2.2.1 Motivations

We first claim that in $\mathbb{R}^3$, the following identity (when applied on a vector field) holds

$$\Delta = \nabla \operatorname{div} - \nabla \times \nabla \times . \tag{2.11}$$

Indeed, for any vector field $u : \mathbb{R}^3 \to \mathbb{R}^3$, for any j,

$$\begin{aligned} \Delta u^j &= (\sum_{k=1}^3 \partial_{x_k x_k} u^j) - \partial_{x_j}(\sum_{k=1}^3 \partial_{x_k} u^k) + \partial_{x_j}(\sum_{k=1}^3 \partial_{x_k} u^k) \\ &= \sum_{k=1}^3 \partial_{x_k}(\partial_{x_k} u^j - \partial_{x_j} u^k) + \partial_{x_j}(\operatorname{div} u). \end{aligned}$$

[8]The students are required to understand the ideas, but not the analysis detail, which is not the focus of the lecture.

In terms of $\omega = \nabla \times u \in \mathbb{R}^3$,

$$\begin{aligned}
\Delta u^1 &= \partial_{x_2}(-\omega^3) + \partial_{x_3}(\omega^2) + \partial_{x_1}(\operatorname{div} u),\\
\Delta u^2 &= \partial_{x_1}(\omega^3) + \partial_{x_3}(-\omega^1) + \partial_{x_2}(\operatorname{div} u),\\
\Delta u^3 &= \partial_{x_1}(-\omega^2) + \partial_{x_2}(\omega^1) + \partial_{x_3}(\operatorname{div} u),
\end{aligned}$$

and hence (2.11) follows. Notice that if the velocity field is divergence-free $\operatorname{div} u = 0$, then u is related to its vorticity $\omega = \nabla \times u$ as follows:

$$-\Delta u = \nabla \times \omega. \tag{2.12}$$

If we could solve the Poisson equation

$$-\Delta v = f,$$

with the solution denoted by $v = (-\Delta)^{-1} f$, then one can recover u from ω as

$$u = (-\Delta)^{-1} \nabla \times \omega.$$

2.2.2 Fundamental solution

Recall the fundamental solution to the Laplace-equation $\Delta v = 0$:

$$\Gamma(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| & N = 2, \\ \frac{1}{(N-2)c_N} |x|^{-(N-2)} & N \geqslant 3, \end{cases} \tag{2.13}$$

where $c_N = |\partial B_1(0)|$ denotes the volume of the unit sphere in $\mathbb{R}^N$. We will show that the Newton potential $\Gamma * f$ solves the Poisson equation $-\Delta v = f$. One can simply calculate (**Exercise**)

$$\begin{aligned}
&\partial_{x_j} \Gamma = g_j \text{ for } x \neq 0,\\
&\partial_{x_i x_j} \Gamma = g_{ij} \text{ for } x \neq 0,\\
&\Delta \Gamma(x) = 0 \text{ for } x \neq 0.
\end{aligned} \tag{2.14}$$

Here

$$\begin{aligned}
g_j(x) &:= -\frac{1}{c_N} \frac{x_j}{|x|^N},\\
g_{ij}(x) &:= -\frac{1}{c_N} \Big(\frac{1}{|x|^N} \delta_{ij} - N \frac{x_i x_j}{|x|^{N+2}} \Big),
\end{aligned} \tag{2.15}$$

and hence

$$\Gamma,\ g_j \in L^1_{\mathrm{loc}}(\mathbb{R}^N), \text{ while } g_{ij} \notin L^1_{\mathrm{loc}}(\mathbb{R}^N), \quad g_{ij} \in L^1_{\mathrm{loc}}(\mathbb{R}^N \backslash \{0\}).$$

Here $L^1_{\text{loc}}(\Omega)$ with $\Omega \subset \mathbb{R}^N$ an open set consists of all (Lebesgue-)measurable function $g : \Omega \to \mathbb{R}$ such that $g \in L^1(K)$ for all compact subset $K \subset \Omega$.
The question we keep in mind is: Is it true that for all $x \in \mathbb{R}^N$,

$$\partial_{x_j}\Gamma = g_j, \quad \partial_{x_i x_j}\Gamma = g_{ij}?$$

2.2.3 Functional spaces & Differentiation

We summerize what we have learned from analysis lectures concerning the differentiation.

1. If $f \in C^1(\mathbb{R}^N)$, then $\partial_{x_j} f \in C(\mathbb{R}^N)$ is well-defined as the limit of $\lim_{h\to 0} \frac{f(x_j+h)-f(x_j)}{h}$, e.g. $f(x) = \sin(x)$ has derivative $f'(x) = \cos(x)$.
2. If $f \in W^{1,p}(\mathbb{R}^N)$, then $\partial_{x_j} f \in L^p(\mathbb{R}^N)$ is the weak derivative of f (see Definition 2.5 below), e.g. $f(x) = \begin{cases} 1+x, & x \in (-1,0] \\ 1-x, & x \in [0,1) \end{cases}$ has weak derivative $f'(x) = \begin{cases} 1, & x \in (-1,0] \\ -1, & x \in (0,1) \end{cases}$
3. If $f \in \mathcal{D}'(\mathbb{R}^N)$ is a distribution, then $\partial_{x_j} f \in \mathcal{D}'(\mathbb{R}^N)$ is the distribution derivative (see Definition 2.6), e.g. the Heaviside function $H(x) = \begin{cases} 0, & x \in (-\infty,0] \\ 1, & x \in (0,1) \end{cases}$ has the distribution derivative $H'(x) = \delta$, where $\langle \delta, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \varphi(0)$.

Definition 2.5 (Lebesgue spaces and Sobolev spaces)**.** *Let $(\Omega, \mathcal{A}, m)$ be a Lebesgue measure space, where $\Omega \subset \mathbb{R}^N$ is an open set, $\mathcal{A}$ consists of Lebesgue-measurable sets restricted in Ω, and m is the Lebesgue measure restricted on Ω.*
Let $1 \leqslant p < \infty$. We call a real-valued Lebesgue-measurable function $f : \Omega \to \overline{\mathbb{R}}$ (i.e. $f^{-1}((t,\infty]) \in \mathcal{A}$ for all $t \in \mathbb{R}$) p integrable if $|f|^p$ is integrable and denote (we denote dm simply by dx from now on)

$$\|f\|_{L^p} = \left(\int_\Omega |f|^p dx \right)^{1/p} = \left(\int_0^\infty m((|f|^p)^{-1}((t,\infty]))dt \right)^{1/p}.$$

We call a Lebesgue-measurable function ∞ integrable or essentially bounded if there is a constant C so that

$$m(\{x : |f(x)| > C\}) = 0.$$

The best constant is denoted by $\|f\|_{L^\infty}$.

We call two measurable functions equivalent (denoted by $f \sim g$), if they are the same almost everywhere (i.e. $m(\{x \in \Omega | f(x) \neq g(x)\}) = 0$).
We define $L^p(\Omega)$ as the set of equivalence classes of p integrable functions.
We define $L^p_{\text{loc}}(\Omega)$ as the set of equivalence classes of p locally integrable functions, which are p integrable on any compact subset of Ω.
Let $f \in L^p(\Omega)$, and we say $f \in W^{1,p}(\Omega)$ if for any $1 \leqslant j \leqslant N$, there exists $h_j \in L^p(\Omega)$ such that

$$\int_\Omega h_j \varphi \, \mathrm{d}x = -\int_\Omega f \partial_{x_j} \varphi \, \mathrm{d}x , \quad \forall \varphi \in C_c^\infty(\Omega).$$

We call h_j the weak derivative of f, and we write simply $h_j = \partial_{x_j} f$.

Definition 2.6 (Distributional derivative). *Let $\mathcal{D}(\mathbb{R}^N) = C_c^\infty(\mathbb{R}^N)$ be the test function space. The distribution space $\mathcal{D}'(\mathbb{R}^N)$ consists of all continuous linear map on $\mathcal{D}(\mathbb{R}^N)$[9]. Let $T \in \mathcal{D}'(\mathbb{R}^N)$, then its (distributional) derivative $\partial_{x_j} T$ is well-defined as a distribution as follows*

$$\langle \partial_{x_j} T, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \langle T, -\partial_{x_j} \varphi \rangle_{\mathcal{D}',\mathcal{D}}, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N).$$

2.2.4 Derivatives of Γ

Any locally integrable function $K \in L^1_{\text{loc}}(\mathbb{R}^N)$ is identified as a distribution $T_K \in \mathcal{D}'$

$$\langle K, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \langle T_K, \varphi \rangle_{\mathcal{D}',\mathcal{D}} = \int_{\mathbb{R}^N} K \varphi dx, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N),$$

and with an abuse of notation we do not distinguish between K and T_K. If furthermore $\partial_j K \in C^1(\mathbb{R}^N \setminus \{0\})$ (not necessarily in $L^1_{\text{loc}}(\mathbb{R}^N)$), then by Gauss' integration formula

$$\begin{aligned} \langle \partial_{x_j} K, \varphi \rangle_{\mathcal{D}',\mathcal{D}} &= -\int_{\mathbb{R}^N} K \partial_{x_j} \varphi dx = -\lim_{\varepsilon \to 0} \int_{\{|x| \geqslant \varepsilon\}} K \partial_{x_j} \varphi dx \\ &= -\lim_{\varepsilon \to 0} \int_{\{|x| \geqslant \varepsilon\}} \big(\partial_{x_j}(K\varphi) - \partial_{x_j} K \varphi\big) dx \qquad (2.16) \\ &= \lim_{\varepsilon \to 0} \Big(\int_{\{|x| \geqslant \varepsilon\}} \partial_{x_j} K \varphi \, \mathrm{d}x + \int_{\{|x| = \varepsilon\}} K \varphi \frac{x_j}{|x|} d\sigma \Big) \\ &= \lim_{\varepsilon \to 0} \Big(\int_{\{|x| \geqslant \varepsilon\}} \partial_{x_j} K \varphi \, \mathrm{d}x \Big) + \lim_{\varepsilon \to 0} \int_{\{|x| = \varepsilon\}} K \varphi \frac{x_j}{|x|} d\sigma. \end{aligned}$$

[08.05.2023]

[15.05.2023]

[9]See e.g. Section 4.2, my notes on Functional Analysis for more details.

Lemma 2.7. *Let* Γ *be given in* (2.13)*, and let* g_j, g_{ij} *be given in* (2.15)*. Then in the distribution sense,*

$$\partial_{x_j}\Gamma = g_j, \tag{2.17}$$

$$\partial_{x_i x_j}\Gamma = \text{p.v.}\, g_{ij} - \frac{1}{N}\delta_{ij}\delta, \tag{2.18}$$

and in particular,

$$-\Delta\Gamma = \delta, \tag{2.19}$$

where $\delta \in \mathcal{D}'(\mathbb{R}^N)$ *denotes the Dirac function:* $\langle \delta, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \varphi(0)$*.*
Here p.v. $g_{ij} \in \mathcal{D}'$ *in* (2.18) *is understood in the sense of Cauchy principle-value integral*

$$\begin{aligned} \langle \text{p.v.}\, g_{ij}, \varphi\rangle_{\mathcal{D}',\mathcal{D}} &= \text{p.v.} \int_{\mathbb{R}^N} g_{ij}\varphi \,\mathrm{d}x := \lim_{\varepsilon\to 0}\int_{\{|x|\geqslant\varepsilon\}} g_{ij}\varphi\,\mathrm{d}x \\ &= \int_{B_1(0)} g_{ij}(x)\big(\varphi(x)-\varphi(0)\big)\,\mathrm{d}x + \int_{(B_1(0))^C} g_{ij}\varphi\,\mathrm{d}x\,. \end{aligned} \tag{2.20}$$

We notice that (2.20) is well-defined: The first integral on the right-hand side makes sense since the integrand is bounded by the following $L^1_{\text{loc}}(\mathbb{R}^N)$-function

$$C\frac{1}{|x|^N}\|\varphi\|_{\text{Lip}}|x| = C\|\varphi\|_{\text{Lip}}|x|^{1-N}$$

and the second integral on the right-hand side is also finite since φ has compact support.

Proof. We first check (2.17): By (2.16),

$$\begin{aligned} \langle \partial_{x_j}\Gamma, \varphi\rangle_{\mathcal{D}',\mathcal{D}} &= \lim_{\varepsilon\to 0}\Big(\int_{\{|x|\geqslant\varepsilon\}} g_j\varphi\,\mathrm{d}x\Big) + \lim_{\varepsilon\to 0}\int_{\{|x|=\varepsilon\}} \Gamma\varphi\,\frac{x_j}{|x|}d\sigma \\ &= \int_{\mathbb{R}^N} g_j\varphi\,\mathrm{d}x + \lim_{\varepsilon\to 0}\int_{\{|y|=1\}}\left(\begin{cases} -\frac{1}{2\pi}\ln|\varepsilon y| & N=2 \\ \frac{1}{(N-2)c_N}|\varepsilon y|^{-(N-2)} & N\geqslant 3\end{cases}\right)\varphi(\varepsilon y)\,\frac{y_j}{|y|}\varepsilon^{N-1}d\sigma \\ &= \langle g_j, \varphi\rangle_{\mathcal{D}',\mathcal{D}}. \end{aligned}$$

We now calculate the distribution $\partial_{x_i}\partial_{x_j}\Gamma = \partial_{x_j}g_i$: We apply (2.16) to $K = g_i = -\frac{1}{c_N}\frac{x_i}{|x|^N}$, where $g_i \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\partial_j g_i \in C^1(\mathbb{R}^N\backslash\{0\}) \subset L^1_{\text{loc}}(\mathbb{R}^N\backslash\{0\})$:

$$\langle \partial_{x_j}g_i, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \lim_{\varepsilon\to 0}\Big(\int_{\{|x|\geqslant\varepsilon\}} g_{ij}\varphi\,\mathrm{d}x\Big) - \frac{1}{c_N}\lim_{\varepsilon\to 0}\int_{\{|x|=\varepsilon\}}\frac{x_i}{\varepsilon^N}\varphi\,\frac{x_j}{\varepsilon}d\sigma$$

where

- the second term on the right-hand side reads

$$-\frac{1}{c_N}\Big(\int_{|x|=1} x_i x_j d\sigma\Big)\varphi(0),$$

which

  - vanishes, if $i \neq j$, since $x_i x_j$ is odd under a reflection $-x_i x_j$;
  - is, if $i = j$,

$$-\frac{1}{c_N}\Big(\int_{|x|=1} x_j^2 d\sigma\Big)\varphi(0) = -\frac{1}{c_N}\Big(\frac{1}{N}\sum_{j=1}^{N}\Big)\Big(\int_{|x|=1} x_j^2 d\sigma\Big)\varphi(0) = -\frac{1}{N}\varphi(0).$$

- the first term on the right-hand side is denoted as Cauchy principle-value integral

$$\text{p.v.} \int_{\mathbb{R}^N} g_{ij}\varphi \,\mathrm{d}x \;= -\frac{1}{c_N}\text{p.v.} \int_{\mathbb{R}^N} \Big(\frac{\delta_{ij}}{|x|^N} - N\frac{x_i x_j}{|x|^{N+2}}\Big)\varphi \,\mathrm{d}x\,.$$

  It is understood as in (2.20), since (by the above argument)

$$\int_{\{|x|=r>0\}} g_{ij} d\sigma = -\frac{1}{c_N}\int_{\{|x|=r>0\}} \Big(\frac{\delta_{ij}}{|x|^N} - N\frac{x_i x_j}{|x|^{N+2}}\Big) d\sigma = 0.$$

Hence (2.20) follows, and in particular,

$$\langle \Delta\Gamma, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \sum_{j=1}^{N}\langle \partial_{x_j} g_j, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = -\varphi(0) = -\langle \delta, \varphi\rangle_{\mathcal{D}',\mathcal{D}}, \qquad \text{i.e.} \;\; -\Delta\Gamma = \delta.$$

□

2.2.5 Newtonian potential

We have the following fact from elliptic theory (this is covered in the lectures "Classical Methods to PDEs" and "Harmonic Analysis"), which we sketch also here by use of Lemma 2.7. The assumptions on f below can be relaxed.

Convolution We recall first the definitions of convolution. For any two test functions $\varphi, \psi \in \mathcal{D}(\mathbb{R}^N)$, we can easily define their convolution $\varphi * \psi \in \mathcal{D}(\mathbb{R}^N)$ by

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^N} \varphi(x-y)\psi(y)\,\mathrm{d}y\,.$$

We can generalize the definition to the convolution between one distribution $T \in \mathcal{D}'(\mathbb{R}^N)$ and one test function $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$(\varphi * T)(x) = \langle T, \varphi(x - \cdot)\rangle_{\mathcal{D}',\mathcal{D}} \in C^\infty(\mathbb{R}^N),$$

or even between one distribution $T \in \mathcal{D}'(\mathbb{R}^N)$ and one distribution with compact support $S \in \mathcal{E}'(\mathbb{R}^N)$:

$$\langle T * S, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \tilde{S} * \varphi\rangle_{\mathcal{D}',\mathcal{D}},$$

where $\tilde{S} \in \mathcal{E}' \subset \mathcal{D}'$ is defined as $\langle \tilde{S}, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \langle S, \varphi(-\cdot)\rangle_{\mathcal{D}',\mathcal{D}}$, such that $\tilde{S} * \varphi \in \mathcal{D}$. For example, the Dirac function $\delta \in \mathcal{D}'$ has compact support $\{0\}$ and hence belongs to $\mathcal{E}'$. In particular

$$T * \delta = T, \quad \forall T \in \mathcal{D}'. \tag{2.21}$$

It is also well known that the convolution can be defined between $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, with $\frac{1}{p} + \frac{1}{q} \geqslant 1$ such that (by Young's inequality)

$$f * g = \int_{\mathbb{R}^N} f(x-y)g(y)\,\mathrm{d}y \in L^r(\mathbb{R}^N), \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{2.22}$$

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Now we can state that the Newtonian potential, as the convolution of the fundamental solution and the source term, is a solution of the Poisson equation.

Lemma 2.8. *Let $f \in L^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ and if $N = 2$, $\int_{\{|x|\geqslant 1\}} |f(x)| \ln |x| dx < \infty$. Then the Newtonian potential*

$$v(x) = (\Gamma * f)(x) := \int_{\mathbb{R}^N} \Gamma(x-y) f(y) dy \in C^2, \tag{2.23}$$

and satisfies

- $$(\nabla v)(x) = (\nabla \Gamma * f)(x) = -\frac{1}{c_N} \int_{\mathbb{R}^N} \frac{x-y}{|x-y|^N} f(y) dy, \tag{2.24}$$

- $$\partial_{ij} v = (\mathrm{p.v.}\, g_{ij}) * f - \frac{1}{N} f \delta_{ij}, \tag{2.25}$$

- *the Poisson equation*

$$-\Delta v = f.$$

The convolution above is understood in the sense of Cauchy principle value integral

$$(\text{p.v.}\, g_{ij}) * f(x) = \text{p.v.} \int_{\mathbb{R}^N} g_{ij}(x-y) f(y)\,\mathrm{d}y$$
$$:= \lim_{\varepsilon \to 0} \int_{\{|x-y| \geqslant \varepsilon\}} g_{ij}(x-y) f(y)\,\mathrm{d}y\,.$$

Proof. If $f \in \mathcal{D}$, the Newtonian potential defined in (2.23)

$$v = \Gamma * f$$

has a distributional derivative

$$\partial_{x_j} v = \partial_{x_j} \langle \Gamma, f(x-\cdot) \rangle_{\mathcal{D}',\mathcal{D}} = \langle \Gamma(y), \partial_{x_j} f(x-y) \rangle_{\mathcal{D}'_y,\mathcal{D}_y}$$
$$= \langle \Gamma(y), -\partial_{y_j} f(x-y) \rangle_{\mathcal{D}'_y,\mathcal{D}_y} = \langle \partial_{y_j} \Gamma(y), f(x-y) \rangle_{\mathcal{D}'_y,\mathcal{D}_y} = \partial_j \Gamma * f$$

which can be represented by (2.24) since $\partial_j \Gamma = g_j \in L^1_{\text{loc}}(\mathbb{R}^N)$. Similarly, one can write $\partial_{ij} v$ as in (2.25):

$$\langle \partial_{ij} \Gamma, f(x-\cdot) \rangle_{\mathcal{D}',\mathcal{D}} = \text{p.v.} \int_{\mathbb{R}^N} g_{ij}(x-y) f(y)\,\mathrm{d}y \; - \frac{1}{N} \delta_{ij} f(x),$$

and hence $-\Delta v = f$.

If f is smooth and sufficiently decaying at infinity as in the assumption, then the integrals (2.23), (2.24) and (2.25) make sense, and hold true (e.g. by density argument). In particular, the Cauchy principle value integral makes sense if $f \in L^1 \cap C^1$ [10] since

$$|(\text{p.v.}\, g_{ij}) * f(x)| = |\text{p.v.} \int_{\mathbb{R}^N} g_{ij}(x-y) \big(f(y) - f(x)\big) dy|$$
$$\leqslant C_1 \int_{\{|x-y| \leqslant 1\}} \frac{1}{|x-y|^N} |f(y) - f(x)| dy + C_1 \int_{\{|x-y| \geqslant 1\}} \frac{1}{|x-y|^N} |f(y)| dy$$
$$\leqslant C_2 \|f\|_{C^1_b} \int_{\{|x-y| \leqslant 1\}} \frac{1}{|x-y|^N} |x-y| dy + C_2 \|f\|_{L^1} < \infty,$$

and hence we can write

$$\text{p.v.} \int_{\mathbb{R}^N} g_{ij}(x-y) f(y)\,\mathrm{d}y \; = \int_{B_1(x)} g_{ij}(x-y) \big(f(y) - f(x)\big) dy$$

[10] Indeed C^α, $\alpha \in (0,1)$ is enough.

$$+\int_{(B_1(x))^C} g_{ij}(x-y)f(y)dy.$$

One can derive from (2.23)-(2.25) that $v \in C^2$ (indeed $v \in C^{2,\alpha}$). **(Exercises)** Thus $-\Delta v = f$ holds in the classical sense. □

2.2.6 Biot-Savart's law in $3D$

By virtue of (2.12) and Lemma 2.8, we have the celebrated Biot-Savart's law. The assumption on ω can be relaxed.

Theorem 2.9 (Biot-Savart's law in $3D$)**.** *If the divergence-free velocity field $u(x) \in L^2(\mathbb{R}^3)$ and its vorticity $\omega(x) = \operatorname{curl}(u(x)) \in \mathbb{R}^3$ are regular and decaying sufficiently fast (e.g. $\omega \in C^1 \cap L^1$), then $u(x)$ can be represented by $\omega(x)$ by*

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\omega(y)}{|x-y|^3}dy. \tag{2.26}$$

Furthermore, ∇u is

$$\begin{aligned}\nabla u(x)h = \text{p.v.}\int_{\mathbb{R}^3}\Big(\frac{1}{4\pi}\frac{\omega(y)\times h}{|x-y|^3} + \frac{3}{4\pi}\frac{[((x-y)\times\omega(y))\otimes(x-y)]h}{|x-y|^5}\Big)dy \\ + \frac{1}{3}\omega(x)\times h, \quad \forall h \in \mathbb{R}^3.\end{aligned} \tag{2.27}$$

Proof. It is straightforward to show that the vector field given by (2.26), denoted from now on by $\tilde{u}$, solves the equation (2.12): $-\Delta u = \nabla\times\omega$. Indeed, we apply Lemma 2.8 to (2.12) (noticing $\frac{1}{4\pi}\frac{x}{|x|^3} = -\nabla\Gamma$) to derive that **(Exercise)**

$$\tilde{u} := -K_3 * \omega, \text{ where the matrix } (K_3) \text{ is given by } K_3(x)h = \frac{1}{4\pi}\frac{x\times h}{|x|^3}, \quad \forall h \in \mathbb{R}^3$$

is a C^1-solution of (2.12), and satisfies (2.27).
One still has to show the uniqueness of the solution to the equation (2.12): $-\Delta u = \nabla\times\omega$ in $L^2(\mathbb{R}^3)$. It is straightforward to derive from Young's inequality (2.22) that **(Exercise)** if $\omega \in L^1 \cap L^\infty$, then (by dividing the integrals near 0 and near ∞ separately)

$$\tilde{u} \in L^r(\mathbb{R}^N), \quad r \in (\frac{3}{2}, \infty]. \tag{2.28}$$

Since $u \in L^2(\mathbb{R}^3)$ satisfies (2.12) in the distribution sense, the difference $\dot{u} := u - \tilde{u} \in L^2(\mathbb{R}^3)$ satisfies the Laplace equation

$$\Delta\dot{u} = 0$$

in the distribution sense. Since any harmonic tempered distribution is polynomial [11], we have $\dot{u} = 0$.
Thus (2.26) holds for $u \in L^2$, and indeed $u \in C^1 \cap L^r$, $r > \frac{3}{2}$, such that its derivatives read as in (2.27). □

The operator from ω to ∇u given by (2.27) is indeed a Calderon-Zygmund operator, which is singular integral operator. We recall the L^p-estimates without proof here.

Lemma 2.10 (L^p-Estimates). *If $\omega \in L^p$ with $p \in (1, \infty)$, then $\nabla u \in L^p$: There exists a constant $C > 0$ such that*

$$\|\nabla u\|_{L^p} \leqslant C \frac{p^2}{p-1} \|\omega\|_{L^p}.$$

Remark 2.11. *By Lemma 2.10,* (2.26) *and* (2.27) *hold for e.g. $\omega \in L^1 \cap L^\infty$. Recall in Example 2.3, the velocity/vorticity field is smooth, but does not decay at infinity, and the Biot-Savart's law does not hold in these cases.*

We take the trace of (2.2) to arrive at another Poisson equation for Π (**Easy exercise.**):

$$-\Delta \Pi = \mathrm{tr}(\nabla u)^2,$$

since $\operatorname{div} u = 0$. Hence one can recover the solution to (2.1) by the solution to (2.5)-(2.26):

Corollary 2.12 (Pressure formula). *If $\omega(t, x) \in \mathbb{R}^3$ is smooth and decaying sufficiently at infinity (e.g. $\omega \in C([0, \infty); L^1 \cap L^\infty)$), and satisfies the equation* (2.5)*: $\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u$ (in the distribution sense) with $u(t, x) \in \mathbb{R}^3$ given by* (2.26)*, then $u(t, x)$ together with*

$$(\nabla \Pi)(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \mathrm{tr}(\nabla u)^2(t, y) dy = \nabla \Gamma * \mathrm{tr}(\nabla u)^2. \tag{2.29}$$

solves (2.1).

Proof. Firstly $u = -K_3 * \omega$ given by (2.26) is divergence-free. Indeed, the identity

$$\Delta u = \nabla \operatorname{div} u - \nabla \times \nabla \times u$$

[11] Since the Fourier transform of a harmonic tempered distribution is supported on the origin, and hence is a linear combination of Dirac function and its derivatives, whose (inverse) Fourier transform is polynomial.

and (2.26)

$$-\Delta u = \nabla \times \omega = \nabla \times \nabla \times u$$

imply

$$\nabla \mathrm{div}\, u = 0,$$

and hence $\mathrm{div}\, u$ is a constant, which is 0 if $\omega \in L^1 \cap L^\infty$, since then by Lemma 2.10 $\nabla u \in L^p$, $p \in (1, \infty)$.

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Since by $\mathrm{div}\, u = 0$ and the equation (2.5) (with ∂_t understood as the distributional derivative)

$$\nabla \times (\partial_t u + u \cdot \nabla u) = \partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u = 0,$$

we have

$$\Delta(\partial_t u + u \cdot \nabla u) = \nabla \mathrm{div}\,(\partial_t u + u \cdot \nabla u) = \nabla\, \mathrm{tr}(\nabla u)^2.$$

Thus by (2.29)

$$\Delta(\partial_t u + u \cdot \nabla u + \nabla \Pi) = 0,$$

and hence the tempered distribution

$$\partial_t u + u \cdot \nabla u + \nabla \Pi$$

vanishes since it decays at infinity by virtue of the following estimates in x-variable:

$$\omega \in L^p, \forall p \in (1, \infty) \Rightarrow u \in L^r, \forall r \in (\frac{3}{2}, \infty] \& \nabla u \in L^p, \forall p \in (1, \infty)$$

$$\Rightarrow u \cdot \nabla u, \mathrm{tr}(\nabla u)^2 \in L^p, \forall p \in (1, \infty) \Rightarrow \nabla \Pi = \nabla \Gamma * \mathrm{tr}(\nabla u)^2 \in L^r, \forall r \in (\frac{3}{2}, \infty].$$

□

Remark 2.13. *Recall the solution* (2.7)*:* $\omega(t, X(t,y)) = (\omega_0 \cdot \nabla) X(t,y)$ *of* (2.5)*. One can rewrite* (2.1) *as a single equation for* $X(t,y)$*:*

$$\partial_t X(t,y) (= u(t, X(t,y))) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{X(t,y) - X(t,y')}{|X(t,y) - X(t,y')|^3} \times (\omega_0 \cdot \nabla) X(t,y')\, \mathrm{d}y',$$

where $X(0,y) = y$*.*

2.3 Local-in-time well-posedness

Given (2.29) (for the case $N = 3$), we are motivated to study the modified Euler equations for $u(t,x) : \mathbb{R}^{N+1} \to \mathbb{R}^N$:

$$\partial_t u + u \cdot \nabla u + A(u,u) = 0, \tag{2.30}$$

where the operator A reads as

$$A(v,w) = \nabla\Gamma * \operatorname{tr}(\nabla v \nabla w) = -\frac{1}{c_N}\frac{x}{|x|^N} * (\sum_{i,j} \partial_i v^j \partial_j w^i).$$

We remark that since $\operatorname{div} u = 0$, one can rewrite $\operatorname{tr}(\nabla u)^2 = \sum_{i,j} \partial_i u^j \partial_j u^i$ in the form

$$\operatorname{tr}(\nabla u)^2 = \sum_{i,j} \partial_{ij}(u^i u^j),$$

and hence A in (2.30) can also be rewritten (at formally) as [12]

$$A(v,w) = \sum_{ij} \nabla\Gamma * \partial_{ij}(v^i w^j) = \sum_{ij} \nabla\partial_{ij}\Gamma * (v^i w^j) = \sum_{ij} \Gamma * \nabla\partial_{ij}(v^i w^j). \tag{2.31}$$

We will benefit from these identities to define the term $A(v,w)$ as a sum $A_1 + \cdots + A_5$ in the functional framework $C^{1,\alpha}$ (see (2.32) below). Formally one can check that (2.1) and (2.30) are equivalent (for e.g. smooth and fast decaying solutions and divergence-free initial data).
In the following we will take arbitrary $N \geqslant 2$, and the data/solutions will be defined on the whole space $\mathbb{R}^N$. The main reference is [1].

2.3.1 Hölder continuous spaces

We introduce the Hölder continuous functional spaces $C^{k,\alpha}$, $\alpha \in (0,1)$, where our solutions will stay in[13]. Roughly speaking, $f \in C^{k,\alpha}$ means that f is $(k+\alpha)$-"times" continuously differentiable. We remark that Hölder continuous spaces $C^{k,\alpha}$, $\alpha \in (0,1)$ are more "friendly" than the usual continuously differentiable spaces C^k for some typical PDEs, e.g. one can derive that the Newtonian potential $v = \Gamma * f \in C^{2,\alpha}$ (locally) if $f \in C^\alpha$, but not $v \in C^2$ if $f \in C$ (as we can see from (2.25)).

[12] By the cancellation property of g_{ij} on the sphere, if $u \in C^\alpha(\mathbb{R}^N)$, $\alpha \in (0,1)$, one can write $A(u,u)$ (rigorously) as

$$A(u,u) = \sum_{i,j} \text{p.v.} \int_{\mathbb{R}^N} \nabla g_{ij}(x-y)\big(u^i(x) - u^i(y)\big)\big(u^j(x) - u^j(y)\big)\,\mathrm{d}y\,.$$

[13] The Sobolev functional framework $W^{s,p}$, $s > 1 + \frac{d}{p}$ is also suitable.

Definition 2.14. *Let $\alpha \in (0,1)$. Let $\Omega \subset \mathbb{R}^N$ be an open set. We call a function f (uniformly) Hölder continuous with exponent α in Ω if*

$$[f]_{\alpha;\Omega} := \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty,$$

and f is called locally Hölder continuous in Ω if $[f]_{\alpha;\Omega'} < \infty$ for all compact subsets $\Omega' \subset \Omega$.
The Hölder space $C^\alpha(\overline{\Omega})$ resp. $C^\alpha(\Omega)$ consists of Hölder continuous functions:

$$\begin{aligned} C^\alpha(\overline{\Omega}) &= \{f \in C(\overline{\Omega}) \,|\, [f]_{\alpha;\Omega} < \infty\}, \\ C^\alpha(\Omega) &= \{f \in C(\Omega) \,|\, [f]_{\alpha;\Omega'} < \infty, \quad \forall \Omega' \subset \Omega \text{ compact subsets}\}. \end{aligned}$$

Similarly, for any $k \in \mathbb{N}$, the Hölder spaces $C^{k,\alpha}(\overline{\Omega})$, $C^{k,\alpha}(\Omega)$ are defined by

$$\begin{aligned} C^{k,\alpha}(\overline{\Omega}) &= \{f \in C^k(\overline{\Omega}) \,|\, [D^k f]_{\alpha;\Omega} := \sup_{|\beta|=k} [D^\beta f]_{\alpha;\Omega} < \infty\}, \\ C^{k,\alpha}(\Omega) &= \{f \in C^k(\Omega) \,|\, [D^k f]_{\alpha;\Omega'} < \infty, \quad \forall \Omega' \subset \Omega \text{ compact subsets}\}. \end{aligned}$$

If $k=0$, then $C^\alpha = C^{0,\alpha}$. If $\Omega = \mathbb{R}^N$, with an abuse of notation, we denote

$$C^{k,\alpha} = C^{k,\alpha}(\mathbb{R}^N) = \{f \in C_b^k(\mathbb{R}^N) \,|\, \|f\|_{C^{k,\alpha}} := \|f\|_{C_b^k} + [D^k f]_{\alpha;\mathbb{R}^N} < \infty\}.$$

Lemma 2.15. *Let $\alpha \in (0,1)$. Then $C^{k,\alpha}$ with $k \in \mathbb{N} \cup \{0\}$ is a Banach space. Furthermore, there exists a constant C such that*

$$\begin{aligned} &\|fg\|_{C^{k,\alpha}} \leqslant C\|f\|_{C^{k,\alpha}}\|g\|_{C^{k,\alpha}}, \quad k=0,1, \\ &\|f\circ g\|_{C^{1,\alpha}} \leqslant C(\|f\|_{C^{1,\alpha}}, \|g\|_{C^{1,\alpha}}), \\ &\|f\|_{C^{1,\alpha'}} \leqslant C\|f\|_{C^\alpha}^\theta \|f\|_{C^{1,\alpha}}^{1-\theta}, \quad \alpha' \in (0,\theta),\ \theta = \alpha - \alpha'. \end{aligned}$$

Proof. Exercise. □

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We have shown that the operator A introduced in (2.30) is well-defined on sufficiently smooth and fast decaying functions v, w (see e.g. Lemma 2.8 or by *sharp* Young's inequality $\nabla v \in L^p, \nabla w \in L^q$ implies $A(v,w) \in L^r$ if $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{N} \in [0,1]$).
The following lemma shows that A is also well-defined on divergence-free $C^{1,\alpha}$-vectors.

Lemma 2.16. *Let $\alpha \in (0,1)$. Then the operator*

$$A : C^{1,\alpha}_\sigma(\mathbb{R}^N;\mathbb{R}^N) \times C^{1,\alpha}_\sigma(\mathbb{R}^N;\mathbb{R}^N) \to C^{1,\alpha}(\mathbb{R}^N;\mathbb{R}^N),$$
$$\text{via } A(v,w) = \nabla\Gamma * \operatorname{tr}(\nabla v \nabla w), \quad \nabla\Gamma = -\frac{1}{c_N}\frac{x}{|x|^N}$$

is a bounded bilinear map, where

$$C^{1,\alpha}_\sigma(\mathbb{R}^N;\mathbb{R}^N) = \{u \in C^{1,\alpha}(\mathbb{R}^N;\mathbb{R}^N) \mid \operatorname{div} u = 0\}.$$

Proof. The proof is not trivial, and it will not be included in the exam.
We sketch the ideas by delicate Fourier analysis. We can rewrite the operator A as (simply by noticing formally $\Gamma * = (-\Delta)^{-1}$ and using Einstein summation convention)

$$A(v,w) = \nabla(-\Delta)^{-1}(\partial_i v^j \partial_j w^i),$$

which reads if $\operatorname{div} v = \operatorname{div} w = 0$ as

$$A(v,w) = \nabla(-\Delta)^{-1}\partial_{ij}(v^j w^i).$$

By use of Bony's decomposition for products[14], it can be decomposed into the following five parts

$$\begin{aligned}
A_1(v,w) &= \nabla(-\Delta)^{-1} T_{\partial_i v^j}\partial_j w^i,\\
A_2(v,w) &= \nabla(-\Delta)^{-1} T_{\partial_j w^i}\partial_i v^j,\\
A_3(v,w) &= \nabla(-\Delta)^{-1}(1-\chi(D))\partial_{ij} R(v^j, w^i),\\
A_4(v,w) &= (\tilde\chi\Gamma) * \nabla\partial_{ij}\chi(D)\partial_{ij} R(v^j, w^i),\\
A_5(v,w) &= \nabla\partial_{ij}((1-\tilde\chi)\Gamma) * \chi(D) R(v^j, w^i),
\end{aligned} \tag{2.32}$$

where $\chi(\xi), \tilde\chi(x)$ are smooth cut-off functions near the origin, in the frequency and space respectively.
Roughly speaking, A_1 cares about the low-frequency part of $\partial_i v^j$ while high-frequency part of $\partial_j w^i$, such that

$$\|A_1(v,w)\|_{C^{1,\alpha}} \leqslant C\|\nabla v\|_{L^\infty}\|\nabla w\|_{C^\alpha} \leqslant C\|v\|_{C^{1,\alpha}}\|w\|_{C^{1,\alpha}}.$$

Similarly it holds for A_2. The operator A_3 involves the disjoint comparable high-frequency parts of v^j, w^i, such that e.g.

$$\|A_3(v,w)\|_{C^{1,2\alpha}} \leqslant C\|v\|_{C^{1,\alpha}}\|w\|_{C^{1,\alpha}}.$$

[14] See e.g. Chapter 2, my notes on Fourier Analysis.

The operators A_4, A_5 take care of the comparable frequency parts of v^j, w^i, and we come back to the convolution formulation for $(-\Delta)^{-1}$. To remove the singularities of Γ, which is not integrable at infinity, we locate Γ near the origin in A_4, such that (noticing A_4 is also located in the frequency)

$$\|A_4\|_{C^{1,\alpha}} \leqslant C\|A_4\|_{L^\infty} \leqslant C\|\tilde{\chi}\Gamma\|_{L^1}\|\chi(D)R(v^j, w^i)\|_{L^\infty} \leqslant C\|v\|_{C^{1,\alpha}}\|w\|_{C^{1,\alpha}}.$$

In A_5 the singularity of Γ at infinity is removed by applying all the derivatives on $(1-\tilde{\chi})\Gamma$, which is then integrable, and the same estimate as for A_4 holds for A_5. To conclude, A is a bounded bilinear map. □

2.3.2 Some typical examples of ODEs

We give here some typical examples of ODEs:

- We consider the ODE

  $$\dot{y}(t) = \alpha(t)y(t) + \beta(t)$$

  with initial data y_0, and α, β are given functions. It is straightforward to calculate from the equation that

  $$\frac{d}{dt}(e^{-\int_0^t \alpha(t')}y(t)) = e^{-\int_0^t \alpha(t')}\beta(t),$$

  and hence by the initial data y_0 the ODE is uniquely solvable (globally in time) as follows

  $$y(t) = e^{\int_0^t \alpha(t')}y_0 + \int_0^t e^{\int_{t'}^t \alpha(t'')dt''}\beta(t')dt'.$$

  Moreover, if $y_0, \alpha, \beta \geqslant 0$, then we can easily derive Gronwall's inequality

  $$y(t) \leqslant e^{\int_0^t \alpha(t')}y_0 + \int_0^t e^{\int_{t'}^t \alpha(t'')dt''}\beta(t')dt',$$

  for $y(t)$ which satisfies

  $$\dot{y}(t) \leqslant \alpha(t)y(t) + \beta(t).$$

- If the righthand side is nonlinear in y, e.g.

  $$\dot{y} = y^2 \tag{2.33}$$

  with initial data $y_0 > 0$, then the unique solution reads as

  $$y(t) = \frac{y_0}{1 - ty_0}, \tag{2.34}$$

  which blows up (tends to ∞) as t tends to $\frac{1}{y_0} \in (0, \infty)$.

2.3.3 Local-in-time wellposedness

We are going to see that after taking the $C^{1,\alpha}$-norm with respect to x-variables, the quadratic ODE (2.33) (more precisely, the estimates of type (2.34), see (2.38) below) will appear, and hence the *local-in-time* results will follow. In order to "see the ODEs" with respect to t-variable, the estimates in $C^\alpha, C^{1,\alpha}$-spaces (w.r.t. x-variable) in Lemma 2.15 and 2.16 will play an essential role. In the following, the operator A in (2.30) will be understood as $A_1 + \cdots + A_5$ in (2.32).

Theorem 2.17. *Let $\alpha \in (0,1)$. Then there exists $c > 0$ such that for any initial data $u_0 \in C^{1,\alpha}(\mathbb{R}^N;\mathbb{R}^N)$, the modified Euler equations* (2.30) *has a unique solution $u \in L^\infty([-T,T];C^{1,\alpha}) \cap_{\alpha'\in(0,\alpha)} C([-T,T];C^{1,\alpha'})$ for some $T \geqslant c\|u_0\|_{C^{1,\alpha}}^{-1} > 0$.*

Proof of Theorem 2.17. **Step 1. Construction of a sequence of global-in-time divergence-free approximate solutions.**
Given $u_n = u_n(t,x) \in L^\infty_{\mathrm{loc}}(\mathbb{R};C^{1,\alpha}(\mathbb{R}^3))$, $n \geqslant 0$, we define iteratively u_{n+1} as the solution of the following *linear* transport equation

$$\begin{cases} \partial_t u_{n+1} + u_n \cdot \nabla u_{n+1} + A(u_n,u_n) = 0, \\ u_{n+1}|_{t=0} = u_0, \end{cases} \tag{2.35}$$

where $A = A_1 + \cdots + A_5$ as in (2.32). By Lemma 2.16, the vector-valued function $A_n := A(u_n,u_n)$ belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R};C^{1,\alpha})$.
Let $X_n(t,y)$ be the Lagrangian trajectory associated to the velocity field u_n (recalling (1.16))

$$\begin{cases} \partial_t X_n(t,y) = u_n(t,X_n(t,y)), \\ X_n(t,y)|_{t=0} = y. \end{cases}$$

If $u_n \in L^\infty_{\mathrm{loc}}(\mathbb{R};C^{1,\alpha})$, then $X^{\pm 1}_{n,t} - \mathrm{Id} \in C(\mathbb{R};C^{1,\alpha})$ such that for all $t \geqslant 0$ **(Exercise)**[15]

$$\begin{aligned} &\|\nabla X^{\pm 1}_{n,t}\|_{L^\infty} \leqslant e^{\int_0^t \|\nabla u_n\|_{L^\infty}\,\mathrm{d}t'}, \\ &\|X^{\pm 1}_{n,t} - \mathrm{Id}\,\|_{C^{1,\alpha}} \leqslant e^{C\int_0^t \|u_n\|_{C^{1,\alpha}}\,\mathrm{d}t'}. \end{aligned} \tag{2.36}$$

The transport equations (2.35) read as

$$\begin{cases} \partial_t(u_{n+1}(t,X_n(t,y))) = -A_n(t,X_n(t,y)), \\ u_{n+1}(t,X_n(t,y))|_{t=0} = u_0(y). \end{cases}$$

[15]Hint: We define more generally the trajectory $X(t,t',y)$ of a velocity field u as

$$X(t,t',y) = y + \int_{t'}^t u(t'',X(t'',t',y))\,\mathrm{d}t''.$$

Then $X_t(y) = X(t,0,y)$ and $X_t^{-1}(y) = X(0,t,y)$.

Integration in time gives us

$$u_{n+1}(t, X_n(t,y)) = u_0(y) - \int_0^t A_n(t', X_n(t',y))dt', \quad \forall t \in \mathbb{R},\, y \in \mathbb{R}^N,$$

and equivalently, the solution of (2.35) reads

$$u_{n+1}(t,x) = u_0(X_{n,t}^{-1}(x)) - \int_0^t A_n(t', X_{n,t'}(X_{n,t}^{-1}(x)))dt', \quad \forall t \in \mathbb{R},\, x \in \mathbb{R}^N .$$

[05.06.2023]

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Hence by Lemma 2.15, Lemma 2.16, (2.36) and Gronwall's inequality we have for $t \geqslant 0$ **(Exercise)**

$$\begin{aligned} &\|u_{n+1}\|_{L^\infty([0,t];C^{1,\alpha})} \\ &\leqslant e^{C\int_0^t \|u_n(t')\|_{C^{1,\alpha}}} \|u_0\|_{C^{1,\alpha}} + \int_0^t \|A_n(t',x)\|_{C^{1,\alpha}} e^{C\int_{t'}^t \|u_n\|_{C^{1,\alpha}}}\, \mathrm{d}t' \\ &\leqslant e^{C\int_0^t \|u_n(t')\|_{C^{1,\alpha}}} \|u_0\|_{C^{1,\alpha}} + \int_0^t C\|u_n(t')\|_{C^{1,\alpha}}^2 e^{C\int_{t'}^t \|u_n\|_{C^{1,\alpha}}}\, \mathrm{d}t' . \end{aligned} \tag{2.37}$$

Similar estimate holds for $t \leqslant 0$. Thus $u_{n+1} \in C(\mathbb{R}; C^{1,\alpha}) \cap L^\infty_{\text{loc}}(\mathbb{R}; C^{1,\alpha})$ given above is the unique solution of (2.35).
Let $t \in \mathbb{R}$ such that $2C|t|\|u_0\|_{C^{1,\alpha}} < 1$. By iteration, we have the following uniform estimates for u_n **(Exercise)**:

$$\|u_n(t)\|_{C^{1,\alpha}} \leqslant \frac{\|u_0\|_{C^{1,\alpha}}}{1 - 2C|t|\|u_0\|_{C^{1,\alpha}}}. \tag{2.38}$$

Step 2. Convergence of the sequence in the weaker topology.
Since the equations (2.30) are invariant under the symmetry $(t,u) \mapsto (-t,-u)$, it suffices to consider positive times.
Let us fix $T > 0$ such that $2CT\|u_0\|_{C^{1,\alpha}} < 1$, and the approximate solutions u_n satisfy uniformly the estimate (2.38):

$$\|u_n\|_{L^\infty([0,T];C^{1,\alpha})} \leqslant \frac{\|u_0\|_{C^{1,\alpha}}}{1 - 2CT\|u_0\|_{C^{1,\alpha}}} =: C_0. \tag{2.39}$$

The iterative equations for the differences $U_{n,m} := (u_{n+m} - u_n)$ read as follows

$$\begin{aligned} &(\partial_t + u_{n+m}\cdot\nabla)U_{n+1,m} \\ &= -U_{n,m}\cdot\nabla u_{n+1} - A(U_{n,m}, u_{n+m} + u_n), \end{aligned}$$

which is a linear equation for $U_{n+1,m}$ if $U_{n,m}$ and (u_n) are given. Since there is a spacial derivative in $U_{n,m}\cdot\nabla u_{n+1}$:

$$\|U_{n,m}\cdot\nabla u_{n+1}\|_{C^\alpha_x}\leqslant C\|U_{n,m}\|_{C^\alpha_x}\|\nabla u_{n+1}\|_{C^\alpha_x}\leqslant C\|U_{n,m}\|_{C^\alpha_x}\|u_{n+1}\|_{C^{1,\alpha}_x},$$

it is convenient to work in a (spatially) weaker topology C^α. Similarly as in Lemma 2.16 (nontrivial),

$$\|A(U_{n,m},u_{n+m}+u_n)\|_{C^\alpha_x}\leqslant C\|U_{n,m}\|_{C^\alpha_x}\|u_{n+m}+u_n\|_{C^{1,\alpha}_x},$$

and we have a similar estimate as in (2.37) for $U_{n+1,m}$:

$$\begin{aligned}&\|U_{n+1,m}\|_{L^\infty([0,t];C^\alpha)}\\&\leqslant e^{C\int_0^t\|u_{n+m}\|_{C^{1,\alpha}}}\int_0^t C\|U_{n,m}(t')\|_{C^\alpha}\|(u_{n+1},u_{n+m},u_n)(t')\|_{C^{1,\alpha}}\,\mathrm{d}t'\,.\end{aligned}$$

By induction it follows **(Exercise)**

$$\|U_{n,m}\|_{L^\infty([0,T];C^\alpha)}\leqslant\frac{1}{n!}(1-2CT\|u_0\|_{C^{1,\alpha}})^{-n}\|U_{0,m}\|_{L^\infty([0,T];C^\alpha)}.$$

By use of the uniform estimates (2.39) for $U_{0,m}$, u_n is a Cauchy sequence in $C([0,T];C^\alpha)$, and hence converges to a limit $u\in C([0,T];C^\alpha)$.

Step 3. Passing to the limit in the equations and final check.
By the uniform bound (2.39), the limit u indeed stays in $L^\infty([0,T];C^{1,\alpha})$. By the interpolation inequality in Lemma 2.15, the sequence u_n converges in a stronger topology:

$$\|u_n-u\|_{L^\infty([0,T];C^{1,\alpha'})}\to 0,\quad\forall\alpha'\in(0,\alpha).$$

This suffices to pass the limit in the equations (2.35) **(Exercise)**, and hence $u\in L^\infty([0,T];C^{1,\alpha})$ is a solution of (2.30), in the distribution sense. Here we recall in the distribution theory that as the time differentiation operator is linear, $u_n\to u$ in $\mathcal{D}'$ implies $\partial_t u_n\to\partial_t u$ in $\mathcal{D}'$. Since $u_n\in C(\mathbb{R};C^{1,\alpha})$, the limit $u\in C([0,T];C^{1,\alpha'})$ and take the value u_0 at the initial time.

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It is the unique solution in $L^\infty([0,T];C^{1,\alpha})$. Indeed, if there are two solutions u_1,u_2 in $L^\infty([0,T];C^{1,\alpha})$, then we can proceed as in Step 2 to consider their difference $\delta u:=u_1-u_2$, which satisfies

$$\|\delta u\|_{L^\infty([0,t];C^\alpha)}\leqslant\exp(C\int_0^t\|u_1\|_{C^{1,\alpha}}\,\mathrm{d}t')\int_0^t\|\delta u(t')\|_{C^\alpha}\|(u_1,u_2)(t')\|_{C^{1,\alpha}}\,\mathrm{d}t'\,.$$

Gronwall's inequality implies $\delta u=0$. The uniqueness follows. □

We hence have the following results for the Euler equations (2.1).

Corollary 2.18. *Let $\alpha \in (0,1)$. Then there exists $c > 0$ such that for any initial data $u_0 \in C^{1,\alpha}(\mathbb{R}^N;\mathbb{R}^N)$ with $\operatorname{div} u_0 = 0$, the Euler equations (2.1) has a unique solution $(u,\nabla\Pi) \in L^\infty([-T,T];C^{1,\alpha}) \cap_{\alpha' \in (0,\alpha)} C([-T,T];C^{1,\alpha'})$ for some $T \geqslant c\|u_0\|^{-1}_{C^{1,\alpha}} > 0$.*

Proof. Let u be the unique solution constructed in Theorem 2.17. We claim that $\operatorname{div} u = 0$. Indeed, we apply div to the modified Euler equations (2.30) to arrive at

$$\begin{cases} \partial_t(\operatorname{div} u) + u \cdot \nabla(\operatorname{div} u) + \operatorname{tr}(\nabla u)^2 + \operatorname{div} A(u,u) = 0, \\ \operatorname{div} u_0 = 0, \end{cases}$$

where, by use of $\Delta\Gamma = -\delta$ or formally $\Delta(-\Delta)^{-1} = -1$,

$$\begin{aligned} &\operatorname{tr}(\nabla u)^2 + \operatorname{div} A(u,u) \\ &= T_{\partial_i u^j}\partial_j u^i + T_{\partial_j u^i}\partial_i u^j + R(\partial_i u^j, \partial_j u^i) - \Big(T_{\partial_i u^j}\partial_j u^i + T_{\partial_j u^i}\partial_i u^j + \partial_{ij}R(u^i,u^j)\Big) \\ &= -R(\partial_j \operatorname{div} u, u^j) - R(u^i, \partial_i \operatorname{div} u) - R(\operatorname{div} u, \operatorname{div} u). \end{aligned}$$

This is essentially transport equation for $\operatorname{div} u$ with null initial data, and hence $\operatorname{div} u = 0$ for all the times[16].
We define $\nabla\Pi = \nabla \operatorname{tr}(\nabla u)^2$ such that $A(u,u) = \nabla\Pi$ and hence $(u,\nabla\Pi)$ satisfy (2.1). The uniqueness follows from the uniqueness result in Theorem 2.17. □

2.4 Two-dimensional case

In this section we restrict ourselves in two-dimensional case $N = 2$. The main reference is [1].

2.4.1 Vorticity revisited

If we are in two-dimensional case: $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $u = \begin{pmatrix} u^1(x_1,x_2) \\ u^2(x_1,x_2) \\ 0 \end{pmatrix}$, then as before we decompose ∇u into its symmetric and antisymmetric parts re-

[16] Similarly as in the proof of Theorem 2.17, the following estimate for $\operatorname{div} u$ comes from the estimates for the remainder operator $R(v,w)$

$$\|\operatorname{div} u\|_{C^\alpha} \leqslant e^{\int_0^t \|u\|_{C^{1,\alpha}}} \|\operatorname{div} u_0\|_{C^\alpha}.$$

spectively:

$$\nabla u = \begin{pmatrix} \partial_{x_1}u^1 & \partial_{x_2}u^1 & 0 \\ \partial_{x_1}u^2 & \partial_{x_2}u^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = d + a$$

$$= \begin{pmatrix} \partial_{x_1}u^1 & \frac{1}{2}(\partial_{x_1}u^2 + \partial_{x_2}u^1) & 0 \\ \frac{1}{2}(\partial_{x_1}u^2 + \partial_{x_2}u^1) & \partial_{x_2}u^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \frac{1}{2}(\partial_{x_2}u^1 - \partial_{x_1}u^2) & 0 \\ \frac{1}{2}(\partial_{x_1}u^2 - \partial_{x_2}u^1) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We define the vorticity ω as a scalar function

$$\omega = \partial_{x_1}u^2 - \partial_{x_2}u^1, \tag{2.40}$$

such that $ah = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times h$, $\forall h \in \mathbb{R}^3$. In the following for notational simplicity we will simply take $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in \mathbb{R}^2$, $\omega = \partial_{x_1}u^2 - \partial_{x_2}u^1 \in \mathbb{R}$. It is straightforward to verify that (**Exercise**)

Lemma 2.19. *Let $N = 2$.*

1. *If the velocity field u satisfies* (2.1) *together with some pressure term, then the vorticity $\omega = \partial_{x_1}u^2 - \partial_{x_2}u^1$ satisfies the free-transport equation*

$$\partial_t\omega + u \cdot \nabla\omega = 0. \tag{2.41}$$

2. *If the divergence-free velocity field $u(x) \in \mathbb{R}^2$ and the vorticity $\omega(x) = \partial_{x_1}u^2 - \partial_{x_2}u^1 \in \mathbb{R}$ are smooth and decaying sufficiently fast at infinity, then $u(x)$ can be represented by (Biot-Savart's law)*

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy, \tag{2.42}$$

where $x^\perp := \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$, and ∇u has a simpler form

$$\nabla u(x) = \frac{1}{2\pi} p.v. \int_{\mathbb{R}^2} \frac{\sigma(x-y)}{|x-y|^2}\omega(y)dy + \frac{1}{2}\omega(x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.43}$$

where

$$\sigma(z) = \frac{1}{|z|^2}\begin{pmatrix} 2z_1z_2 & z_2^2 - z_1^2 \\ z_2^2 - z_1^2 & -2z_1z_2 \end{pmatrix}.$$

3. If a smooth and fast decaying function $\omega(t,x)$ solves (2.41), then (2.42) together with $\nabla\Pi(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} tr(\nabla u)^2 dy$ solves (2.1).

Remark 2.20 (Stream function)**.** *Let $\psi \in \mathbb{R}$ be a stream function such that*

$$u = \nabla^\perp \psi, \text{ that is, } \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} -\partial_2\psi \\ \partial_1\psi \end{pmatrix} \tag{2.44}$$

is divergence-free. Then ψ satisfies the Poisson equation with ω as the source term

$$\Delta\psi = \partial_1(\partial_1\psi) + \partial_2(\partial_2\psi) = \partial_1(u^2) + \partial_2(-u^1) = \omega. \tag{2.45}$$

Conversely, if u is divergence-free velocity field which is smooth and fast decaying at infinity, then there exists a stream function $\psi = -(-\Delta)^{-1}\omega$ such that $u = \nabla^\perp\psi$.

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2.4.2 Global-in-time well-posedness in $2D$

The local-in-time wellposedness in any dimension $N \geqslant 2$ has been established in Corollary 2.18, and the prototypical ODE in the proof is (2.33), whose solution (2.34) blows up in finite time. We are going to see that in dimension two, a "linear" ODE will appear (see (2.48)-(2.50) below) thanks to the a priori estimates for the vorticity (see (2.49) below) which satisfies the free transport equation (2.40). Recall the vorticity equation (2.5) for $N = 3$, where there is an additional nonlinear term on the righthand side, and hence the following strategy for dimension two does not work for dimension three.

Theorem 2.21. *Let $N = 2$. Let $u_0 \in C^{1,\alpha}(\mathbb{R}^2)$, $\alpha \in (0,1)$ be a divergence-free vector field, such that $\omega_0 = \nabla^\perp u_0 \in L^1 \cap L^\infty$. Then the Euler equations (2.21) have a unique global-in-time solution $(u, \nabla\Pi)$ such that*

$$u \in L^\infty_{\text{loc}}(\mathbb{R}; C^{1,\alpha}), \quad \omega \in C(\mathbb{R}; L^1 \cap L^\infty).$$

Proof. We sketch the proof ideas here. By symmetry if suffices to consider positive times. The main strategy here is to "play" with the norms with respect to the x-variables (which is impossible for ODEs where only the time variable is present).

Step 1. Continuation criteria. Let $u_0 \in C^{1,\alpha}$ be a divergence-free initial data. Let T^* denote the maximal existence time of the solution

$(u, \nabla\Pi) \in L^\infty_{\mathrm{loc}}([0,T^*); C^{1,\alpha}) \cap_{\alpha'\in(0,\alpha)} C([0,T^*); C^{1,\alpha'})$ for Euler equations (2.1), or equivalently for (2.30). Obviously $T^* > 0$ by Theorem 2.17. We claim that

$$T^* < \infty \Longrightarrow \int_0^{T^*} \|\nabla u\|_{L^\infty} dt = \infty. \tag{2.46}$$

Indeed, by similar arguments implying the estimates (2.37) in Proof of Theorem 2.17, the following more refined a priori estimates [17] hold for solutions of the (modified) Euler equations (2.30): $\partial_t u + u\cdot\nabla u + A(u,u) = 0$:

$$\|u(t)\|_{C^{1,\alpha}} \leqslant \|u_0\|_{C^{1,\alpha}} \exp\Big(C\int_0^t \|\nabla u\|_{L^\infty}\, \mathrm{d}t'\Big). \tag{2.47}$$

Thus if the converse of (2.46) holds

$$\int_0^{T^*} \|\nabla u\|_{L^\infty} dt =: c < \infty,$$

then

$$\|u\|_{L^\infty([0,T^*]; C^{1,\alpha})} \leqslant e^{Cc}\|u_0\|_{C^{1,\alpha}}.$$

For any $\varepsilon < \min\{\frac{1}{2Ce^{Cc}\|u_0\|_{C^{1,\alpha}}}, T^*\}$, by Theorem 2.17 there exists a unique solution

$$u \in L^\infty([T^*-\varepsilon/2, T^*+\varepsilon/2]; C^{1,\alpha}) \cap_{\alpha'\in(0,\alpha)} C([T^*-\varepsilon/2, T^*+\varepsilon/2]; C^{1,\alpha'}).$$

We have extended the solution beyond T^*, which is a contradiction to the maximality of the existence time T^*. Thus (2.46) holds.
Step 2. Refined continuation criteria. We claim that the Lip-norm in (2.46) can be replaced by a weaker Besov-norm $B^1_{\infty,\infty}$:

$$\int_0^{T^*} \|u\|_{B^1_{\infty,\infty}} dt < \infty \Longrightarrow \int_0^{T^*} \|\nabla u\|_{L^\infty} dt < \infty.$$

Indeed, this can be achieved by explore the delicate estimate

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\leqslant C\|u\|_{B^1_{\infty,\infty}} \ln\Big(e + \frac{\|u\|_{C^{1,\alpha}}}{\|u\|_{B^1_{\infty,\infty}}}\Big)\\ &\leqslant C\underbrace{\max\{\|u(t)\|_{B^1_{\infty,\infty}}, \|u_0\|_{B^1_{\infty,\infty}}\}}_{=:U_1(t)} \ln\Big(1 + \frac{\|u\|_{C^{1,\alpha}}}{\|u_0\|_{B^1_{\infty,\infty}}}\Big).\end{aligned}$$

[17] We replace $\|u\|_{C^{1,\alpha}}$−norm in (2.37) by $\|u\|_{\mathrm{Lip}}$−norm, by using e.g. the estimate $\|A(v,w)\|_{C^{1,\alpha}} \leqslant C(\|v\|_{\mathrm{Lip}}\|w\|_{C^{1,\alpha}} + \|v\|_{C^{1,\alpha}}\|w\|_{\mathrm{Lip}})$ instead of Lemma 2.16.

We the estimate (2.47) to it to derive

$$\|\nabla u\|_{L^\infty} \leqslant CU_1 \ln(1 + \frac{\|u_0\|_{C^{1,\alpha}}}{\|u_0\|_{B^1_{\infty,\infty}}})\Big(1 + \int_0^t \|\nabla u\|_{L^\infty}\Big), \tag{2.48}$$

which together with Gronwall's inequality gives

$$\int_0^t \|\nabla u\|_{L^\infty} \leqslant \exp\Big(C \ln(1 + \frac{\|u_0\|_{C^{1,\alpha}}}{\|u_0\|_{B^1_{\infty,\infty}}}) \int_0^t U_1\Big) - 1.$$

Step 3. A priori estimates for the vorticity. Since ω satisfies the free transport equation (2.41), all the L^p-norm of ω is conserved a priori by the volume-preserving flow $X(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\omega(t, X(t, y)) = \omega_0(y) \Longrightarrow \|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}, \quad \forall p \in [1, \infty]. \tag{2.49}$$

Similar as (2.28) in $3D$, Young's inequality implies that for $\omega \in L^1 \cap L^\infty$, u given in (2.42) satisfies

$$u \in L^r, \quad \forall r \in [2, \infty].$$

By use of some Fourier analysis we know that

$$\|u(t)\|_{B^1_{\infty,\infty}} \leqslant C(\|u(t)\|_{L^r} + \|\omega(t)\|_{L^\infty}) \leqslant C\|\omega_0\|_{L^1 \cap L^\infty}. \tag{2.50}$$

This implies that for any finite time $t < \infty$, $\int_0^t \|u\|_{B^1_{\infty,\infty}} < \infty$, and hence $\int_0^t \|\nabla u\|_{L^\infty} < \infty$ by Step 2, and thus $T^* = \infty$ by Step 1. □

2.5 One dimensional isentropic compressible Euler equations

We have discussed until now the incompressible Euler equations (2.1), for higher dimensions $N \geqslant 2$. Notice that if $N = 1$ then $\operatorname{div} u = 0$ reduces to the fact that u is a constant, which is of no interest.

In this subsection some mathematical theory for the one-dimensional isentropic compressible Euler equations is briefly mentioned, i.e. we consider the models (1.15) in the inviscid case $\mu = \lambda = 0$ in dimension $N = 1$ [18]:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) + \partial_x(p(\rho)) = 0, \end{cases} \tag{2.51}$$

where $t, x \in \mathbb{R}$, (ρ, u) is a pair of unknown functions and $p = p(\rho)$ is a given function, e.g. $p(\rho) = \rho^\gamma$, $\gamma > 1$. We can rewrite (2.51) in a first-order system of conservation laws

$$\partial_t v + \partial_x f(v) = 0, \tag{2.52}$$

where

$$v = \begin{pmatrix} \rho \\ u \end{pmatrix}, \quad f(v) = \begin{pmatrix} \rho u \\ \frac{1}{2}u^2 + p_1(\rho) \end{pmatrix} \text{ with } p_1'(z) = \frac{1}{z}p'(z),$$

or equivalently,

$$\partial_t v + a(v)\partial_x v = 0, \text{ with } a(v) := (\nabla_v f) = \begin{pmatrix} u & \rho \\ \frac{p'(\rho)}{\rho} & u \end{pmatrix}. \tag{2.53}$$

The mathematical theory is rather different from the incompressible case, and typical wave phenomena such as rarefraction waves and shock waves are present. The main reference of this section is [2].

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2.5.1 Burgers' equation

As a warmup, we consider $n = 1$ and the Cauchy problem for the celebrated Burgers' (inviscid) equation

$$\partial_t v + v\partial_x v = 0, \quad v(t, x)|_{t=0} = v_0(x). \tag{2.54}$$

It is also called Hopf's equation occationally.

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[18]Sometimes it is convenient to work with the specific volume $v := \frac{1}{\rho}$ (instead of ρ):

$$\begin{cases} \partial_t v + u\partial_x v - v\partial_x u = 0, \\ \partial_t u + u\partial_x u + v\partial_x(p(\rho)) = 0, \end{cases}$$

and in some Lagrangian coordinate (t, y) (nontrivial), (v, u) equations read

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p_2(v) = 0. \end{cases}$$

Classical solutions If $v \in C^1$ is a classical solution, then as usual we define the Lagrangian coordinate $X(t, y)$ by

$$\partial_t X(t, y) = v(t, X(t, y)) \text{ with } X(0) = y,$$

such that

$$v(t, X(t, y)) = v_0(y).$$

This implies that the flow $X(t, y)$ are straight lines:

$$\partial_t X(t, y) = v_0(y) \text{ with } X(0) = y, \text{ i.e. } X(t, y) = y + v_0(y)t.$$

If $X_t : \mathbb{R} \to \mathbb{R}$ is invertible all the times, the solution is given by

$$v(t, x) = v_0(X_t^{-1}(x)).$$

Since the invertibility of $X_t : \mathbb{R} \to \mathbb{R}$ is equivalent to the nonzero of the Jacobian $\det(\partial_y X_t)$[19], it depends heavily on the initial data: Observe that if $v_0 \in C_b^1$, then

$$\partial_y X_t(y) = 1 + v_0'(y)t,$$

which means that

- If $v_0' \geqslant 0$ everywhere, then X_t is globally-in-time invertible and the solution is given by $v(t, x) = v_0(X_t^{-1}(x))$.
- If $v_0'(y_0) = \inf_{\mathbb{R}} v_0' < 0$ at some point $y_0 \in \mathbb{R}$, then $X_t : \mathbb{R} \to \mathbb{R}$ is invertible only up to the time

  $$T^* = -\frac{1}{v_0'(y_0)},$$

  and there are no C^1-solutions beyond the strip $[0, T^*)$: More precisely,

  $$\partial_x X_t^{-1}(x) = \frac{1}{\partial_y X_t(y)}\Big|_{y=X_t^{-1}(x)} = \frac{1}{1 + v_0'(X_t^{-1}(x))t},$$

  $$\partial_x v(t, x) = \partial_y v_0(X_t^{-1}(x)) \partial_x X_t^{-1}(x) = \frac{v_0'(X_t^{-1}(x))}{1 + v_0'(X_t^{-1}(x))t}.$$

  This is typical PDE-type blowup (shock wave) and the derivatives of the solution cease to be bounded, which does not happen in ODEs (compare it to (2.33)-(2.34)).

We conclude in particular if the initial data $v_0 \in C_c^\infty(\mathbb{R})$ has compact support and is not identically zero, then the solution can not exist globally in time, no matter how small it is. We then generalize below the definition of solutions.

[19]We recall that in the incompressible case $\operatorname{div} u = 0$, by virtue of (1.17), the associated flow $X_t : \mathbb{R}^N \to \mathbb{R}^N$ is invertible and volume-preserving, as long as the solution u is existing and is Lipschity continuous w.r.t x-variable (in the local-in-time sense).

Weak solutions It motivates to consider the weak solutions of the Cauchy problem for the Burgers equation in the form of conservation law

$$\begin{cases} \partial_t v + \partial_x(f(v)) = 0, & \text{with } f(v) = \frac{1}{2}v^2, \\ v|_{t=0} = v_0. \end{cases} \tag{2.55}$$

If $v \in C_b^1$ is a classical solution, then we test the above equation by a function $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R})$, and integration by parts implies the following equality

$$\int_0^\infty \int_{\mathbb{R}} \big(v\partial_t\varphi + \frac{1}{2}v^2\partial_x\varphi\big)dxdt + \int_{\mathbb{R}} v_0(x)\varphi(0,x)dx = 0, \quad \forall \varphi \in C_c^\infty([0,\infty) \times \mathbb{R}). \tag{2.56}$$

We call a function $v \in L^2_{\text{loc}} \subset \mathcal{D}'$ weak solution of (2.55), if (2.56) holds, that is, the equation and the initial condition are satisfied in the distribution sense.

If v is a weak solution of (2.55) and $v \in C^1$ on both sides of a C^1-curve $\{x = x(\tau), t = t(\tau) \,|\, \tau \in [a,b]\}$ on the (x,t)-plane which is parametrized by the parameter τ in some interval $[a,b] \in \mathbb{R}$. Show that the slope $c(\tau) := \frac{x'(\tau)}{t'(\tau)}$ of this curve satisfies the Rankine-Hugoniot condition **(Exercise.)**[20]

$$\begin{aligned} c(\tau) &= \frac{f(v_+(\tau)) - f(v_-(\tau))}{v_+(\tau) - v_-(\tau)} \\ &= \frac{1}{2}(v_+(\tau) + v_-(\tau)), \end{aligned} \tag{2.57}$$

where $v_\pm(\tau)$ denote the left and right limits of the solution $v(t,x)$ at the curve respectively. It is interesting to consider the Riemann problem for the Burgers' equation with piecewise-constant initial data **(Exercise. Verify the following. Draw a picture.)**:

1. If $v_0(y) = \begin{cases} 0 & \text{if } y \leqslant 0 \\ 1 & \text{if } y > 0 \end{cases}$, then there is a continuous solution

$$v(t,x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x/t & \text{if } 0 \leqslant x \leqslant t \\ 1 & \text{if } x > t. \end{cases} \tag{2.58}$$

[20]Note that the unit outer normal vector of the curve is $n = \frac{1}{|(x'(\tau),t'(\tau))|}\begin{pmatrix} t'(\tau) \\ -x'(\tau) \end{pmatrix}$ and the conservation law reads in divergence-form $\text{div}_{x,t}\begin{pmatrix} f(v) \\ v \end{pmatrix} = 0$. Gaussian Integral formula implies the jump condition.

2. If $v_0(y) = \begin{cases} 1 & \text{if } y \leqslant 0 \\ 0 & \text{if } y > 0 \end{cases}$, then there is a discontinuous weak solution

$$v(t,x) = \begin{cases} 1 & \text{if } x \leqslant \frac{1}{2}t \\ 0 & \text{if } x > \frac{1}{2}t. \end{cases} \tag{2.59}$$

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In the first case the solution is continuous except the singularity $(0,0)$, and is Lipschitz continuous in $(0,\infty)\times\mathbb{R}$ away from the initial time. Notice that

- The weak solution $v(t,x)$ in (2.58) is self-similar

$$v(t,x) = \phi(s(t,x)), \quad \phi(s) = \begin{cases} 0 & \text{if } s \leqslant 0, \\ s & \text{if } 0 < s < 1, \\ 1 & \text{if } s \geqslant 1, \end{cases} \quad s(t,x) = \frac{x}{t}.$$

- The curve $\{\phi(s) \,|\, s \in [0,1]\}$ is an integral curve of the constant "vector field" 1:

$$\phi'(s) = 1(\phi(s)) = 1,$$

which connects the left status $s = 0 = v_0|_{\mathbb{R}^-}$ and the right status $s = 1 = v_0|_{\mathbb{R}^+}$;

- $\{c = s(t,x) \,|\, t,x \in \mathbb{R}\}$ defines a straight line in (t,x)-plane with slope c, and the solution $v(t,x)$ is constant along this straight line.

Such solution is called rarefaction wave or simple wave.
In the second case the singularity at the origin is propagated along the straight line $\{0 = x - \frac{1}{2}t \,|\, t,x \in \mathbb{R}\}$ where the slope $\frac{1}{2}$ is calculated by the condition (2.57).

2.5.2 One dimensional isentropic compressible Euler equations

We consider the Riemann problem for the one-dimensional compressible Euler equations (2.53):

$$\partial_t v + a(v)\partial_x v = 0, \text{ with } a(v) := (\nabla_v f) = \begin{pmatrix} u & \rho \\ \frac{c^2(\rho)}{\rho} & u \end{pmatrix}. \tag{2.60}$$

where for notational convenience we have introduced (the sound speed) $c = c(\rho)$, defined by

$$c(\rho) = \sqrt{p'(\rho)} > 0.$$

Obviously

$$v = \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a constant solution of (2.60), and in the following *small* initial data or *small* solutions are always understood as small perturbations of the state $(1,0)^T$: This is reasonable since we can introduce $\tilde{\rho} = \rho - 1$ and consider the equations for $\tilde{v} := \begin{pmatrix} \tilde{\rho} \\ u \end{pmatrix}$ with small initial data.

It is straightforward to calculate that the 2×2 matrix $a(v) = \begin{pmatrix} u & \rho \\ \frac{c^2(\rho)}{\rho} & u \end{pmatrix}$ has two distinct eigenvalues

$$\lambda_1(v) = u - c(\rho), \quad \lambda_2(v) = u + c(\rho),$$

and the corresponding eigenvectors could be

$$r_1(v) = \begin{pmatrix} 1 \\ -\frac{c(\rho)}{\rho} \end{pmatrix}, \quad r_2(v) = \begin{pmatrix} 1 \\ \frac{c(\rho)}{\rho} \end{pmatrix}.$$

Remark that for small solutions v, $\lambda_1(v)$ is close to $-c(1) < 0$ while $\lambda_2(v)$ is close to $c(1) > 0$, such that

$$\lambda_1(v) < \lambda_2(v).$$

We calculate $\nabla\lambda_1 = \begin{pmatrix} -c'(\rho) \\ 1 \end{pmatrix}$ and $\nabla\lambda_2 = \begin{pmatrix} c'(\rho) \\ 1 \end{pmatrix}$. There are two different types of nonlinearities:

(N1) If $c'(\rho) + \frac{c(\rho)}{\rho} \neq 0$, then $r_j \cdot \nabla\lambda_j \neq 0$, $j = 1, 2$, and we can normalize $r_1 = -\frac{1}{c'(\rho) + \frac{c(\rho)}{\rho}} \begin{pmatrix} 1 \\ -\frac{c(\rho)}{\rho} \end{pmatrix}$ (similar for r_2) such that

$$r_1 \cdot \nabla\lambda_1 = r_2 \cdot \nabla\lambda_2 = 1. \tag{2.61}$$

We call a nonlinear hyperbolic system of first order satisfying (2.61) genuinely nonlinear.

(N2) If $c'(\rho) + \frac{c(\rho)}{\rho} = 0$, i.e. $c(\rho) = \frac{1}{\rho}$ (up to constants), then

$$r_j \cdot \nabla\lambda_j = 0, \quad j = 1, 2.$$

We call such a nonlinear hyperbolic system of first order totally linearly degenerate.

Recall the Burgers equation (2.54), where the "matrix" $a(v) = v$ has the eigenvalue $\lambda(v) = v$ and the associated (renormalized constant) eigenvector 1. By the above definition it is genuinely nonlinear hyperbolic equation. Heuristically genuine nonlinearity means that the characteristics (associated to the same eigenvalue $\lambda_j(v)$) may interact each other, as presented in the solution (2.59).

Riemann problem We consider the Riemann problem for (2.60) in the form of conservation laws

$$\partial_t v + \partial_x f(v) = 0, \quad f(v) = \begin{pmatrix} \rho u \\ \frac{1}{2}u^2 + p_1(\rho) \end{pmatrix} \text{ with } p_1'(z) = \frac{1}{z}p'(z), \tag{2.62}$$

in the case of genuine nonlinearity (see (2.61) above), equipped with the initial data

$$v_0(x) = \begin{cases} v_- & \text{if } x \leqslant 0, \\ v_+ & \text{if } x > 0, \end{cases} \tag{2.63}$$

where $v_+ \neq v_-$ are two small different vectors in $\mathbb{R}^2$. We call $v(t,x)$ a weak solution of (2.62) with the initial data v_0 if (2.56) holds.
In analogue to the rarefaction wave solution (2.58) and the shock wave solution (2.59) for the Riemann problem of Burgers' equation, we discuss first j-simple rarefaction wave and j-shock wave below.

1. j-simple rarefaction waves. Let $\mathbb{R} \ni s \mapsto \phi(s) \in \mathbb{R}^2$ be the orbit of r_j (in the v-space/manifold) starting from v_-:

   $$\phi'(s) = r_j(\phi(s)), \quad \phi(0) = v_-. \tag{2.64}$$

   Then if $s : \mathbb{R}^2 \to \mathbb{R}$ satisfies (see (2.76) below for more explanations)

   $$\partial_t s + \lambda_j(\phi(s))\partial_x s = 0, \tag{2.65}$$

   then $v(t,x) := \phi(s(t,x))$ solves $\partial_t v + a(v)\partial_x v = 0$, which is called j-simple rarefaction wave.

   Hence if $v_+ = \phi(s^*)$ for some $s^* > 0$ and $v_\pm$ are small enough, then $\phi(s)$ gives a Lipschitz continuous solution of the Riemann problem.

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Furthermore, since (by virtue of (2.61))

$$\frac{d}{ds}\lambda_j(\phi(s)) = \phi'(s) \cdot \nabla_v \lambda_j(\phi(s)) = (r_j \cdot \nabla \lambda_j)(\phi(s)) = 1,$$

$\lambda_j(\phi(\cdot))$ is strictly increasing from 0 to s^*:

$$\begin{aligned} \lambda_j(v_-) = \lambda_j(\phi(0)) < \lambda_j(\phi(s_1)) < \lambda_j(\phi(s_2)) \\ < \lambda_j(\phi(s^*)) = \lambda_j(v_+), \quad \forall 0 < s_1 < s_2 < s^*. \end{aligned} \tag{2.66}$$

2. j-shock waves. If the Riemann problem has a discontinuous solution

$$v(t,x) = v_\pm \text{ for } \pm(x - ct) > 0, \tag{2.67}$$

then the (two) Rankine-Hugoniot conditions for weak solutions of (2.52):

$$c(v_+ - v_-) = f(v_+) - f(v_-)$$

implies one equation for v_+ for given v_-.

Intuitively, if $v_+ \neq v_-$ are both small, then

$$f(v_+)-f(v_-) = \int_0^1 a(v_- + t(v_+ - v_-))dt(v_+ - v_-) =: a(v_+, v_-)(v_+ - v_-),$$

and we define $\lambda_j(v_+, v_-), r_j(v_+, v_-)$ as the corresponding eigenvalues and eigenvectors of $a(v_+, v_-)$. This implies, for some j and some $\sigma \in \mathbb{R}$

$$c = \lambda_j(v_+, v_-), \quad v_+ - v_- = \sigma r_j(v_+, v_-). \tag{2.68}$$

Hence v_+ should satisfy

$$v - v_- - \sigma r_j(v, v_-) = 0$$

for some σ, that is, the above two equations should be satisfied by three unknowns (v_+, σ). Since the Jacobian of the lefthand side w.r.t. v when $\sigma = 0$ is the identity, it follows from the implicit function theorem that v is a smooth function of σ such that $v'(\sigma)|_{\sigma=0} = r_j(v_-)$. Hence such a solution (2.67) exists if and only if v_+ stays in a curve through v_-, which is tangent to the orbit of r_j.

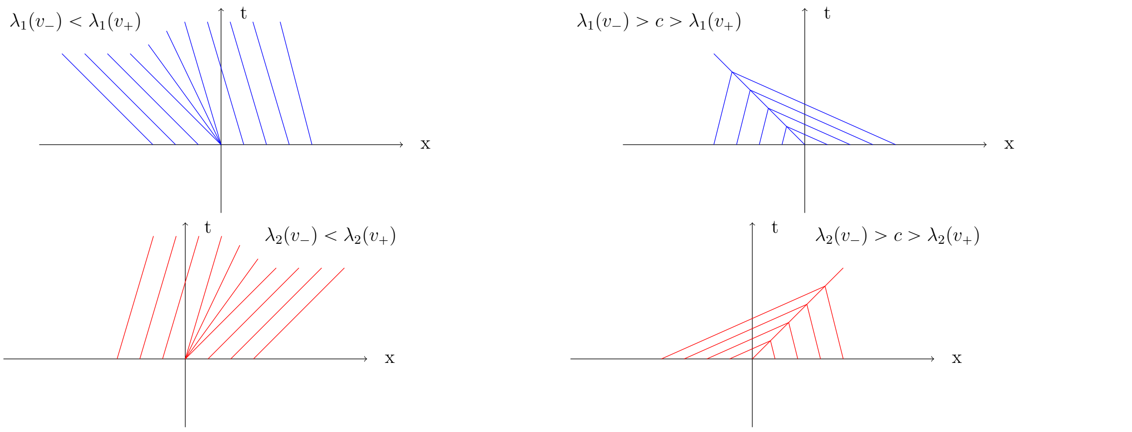
If v_- and v_+ are both small, then up to higher order terms,

$$\lambda_j(v_+) - \lambda_j(v_-) \sim (v_+ - v_-) \cdot \nabla \lambda_j(v_-) \sim \sigma r_j(v_-) \cdot \nabla \lambda_j(v_-).$$

By virtue of the genuinely nonlinear condition $r_j \cdot \nabla\lambda_j = 1 > 0$,

$$\sigma < 0, \text{ and hence } \lambda_j(v_+) < c < \lambda_j(v_-), \tag{2.69}$$

where the case $\sigma > 0$ such that $\lambda_j(v_-) < c = \lambda_j(v_+, v_-) < \lambda_j(v_+)$ does not hold for shock-solution (2.67), since in this case j-characteristics are pointing away from the shock at both sides, which is rejected in favor of a rarefaction wave (see also (2.66)). Such as solution (2.67) satisfying (2.69) is called an *admissible* j-shock wave solution.



(a) Characteristics for a 1-simple rarefaction wave (above) and 2-simple rarefaction wave (below).

(b) Interaction of characteristics for a 1-shock wave (above) and 2-shock wave (below).

Figure 1

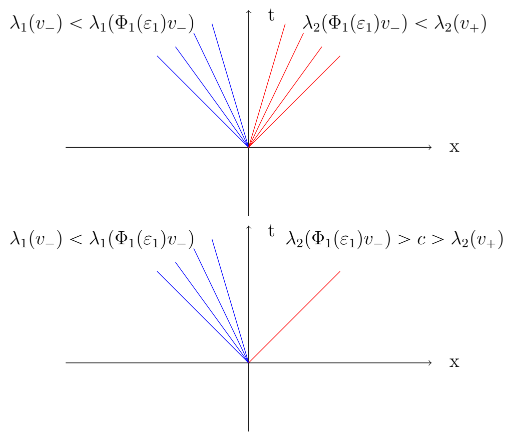
To conclude, for $v_- \in \mathbb{R}^2$ we define $\Phi_j(\varepsilon)$ as

$$\Phi_j(\varepsilon)v_- = v_+,$$

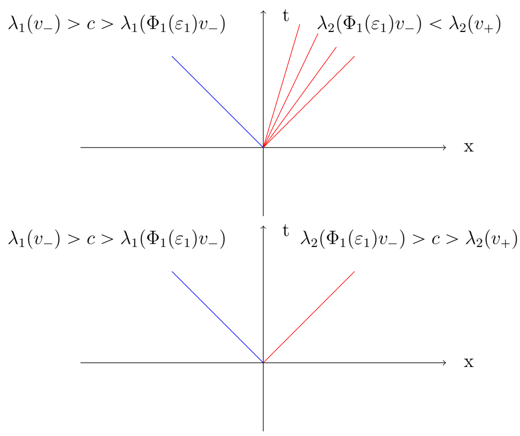
if $\varepsilon \geqslant 0$ and v_+ is the value for $s = \varepsilon$ of the solution (2.64)-(2.65), or $\varepsilon < 0$ and v_+ is the solution (2.67)-(2.68) with $\sigma = \varepsilon$. Thus the Riemann problem with $v_+ = \Phi_j(\varepsilon)v_-$ is solved by a j-simple rarefaction wave if $\varepsilon \geqslant 0$ (see Figure 1a)or by an admissible j-shock wave if $\varepsilon < 0$ (see Figure 1b).
If

$$v_+ = \Phi_2(\varepsilon_2)\Phi_1(\varepsilon_1)v_-, \tag{2.70}$$

for small ε_j, then we have obtained a solution of the Riemann problem (2.62)-(2.63) consisting in order of increasing j from left to right of a j rarefaction wave or an admissible j shock wave of strength ε_j. More precisely, from left to right, v_- is connected to some middle state $\Phi_1(\varepsilon_1)v_-$ by a 1-rarefaction/shock wave of strength ε_1, and then $\Phi_1(\varepsilon_1)v_-$ is connected to v_+ by a 2-rarefaction/shock wave of strength ε_2 (see Figure 2a, Figure 2b).



(a) v_- and v_+ are connected by, from left to right, a 1-simple rarefaction wave (blue) and a 2-simple rarefaction wave (above), or a 2-shock wave (below).



(b) v_- and v_+ are connected by, from left to right, a 1-shock wave (blue) and a 2-simple rarefaction wave (above), or a 2-shock wave (below).

Figure 2

The 1-wave and 2-wave do not interfere with each other since they move from the origin with quite different speeds, close to $\lambda_1 \sim -c(1)$, $\lambda_2 \sim c(1)$, respectively. For small $v_\pm$, the equation (2.70) determines (ε_j) uniquely:

Theorem 2.22 (Unique solvability of the Riemann problem for one-dimensional isentropic compressible Euler equations in the case of genuinely nonlinearity)**.** *If v_-, v_+ are sufficiently small, then the Riemann problem* (2.62)-(2.63) *has a unique solution consisting from left to right for increasing j of a small j simple rarefaction wave or a small j admissible shock, $j = 1, 2$.*

We don't give the proof here, which can be found in [2].
We conclude with more explanations for the two different nonlinearities:

(N1) If (2.60) is genuinely nonlinear, then for small perturbed compactly supported initial data (ρ_0, u_0) (around $(1, 0)$) simple waves and shock waves arise for finite time. This is the case similar as for Burgers' equation. Roughly speaking, if a nonlinear hyperbolic conservation law system is genuinely nonlinear, then the j-characteristics, i.e. the curves $s : (t, x) \ni \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\partial_t s + \lambda_j(v)\partial_x s = 0, \quad j = 1 \text{ or } 2 \tag{2.71}$$

may interact with each other (see (2.69) above) and in particular, when two characteristics encounter each other, shock waves appear and the solution ceases to be classical (see (2.59) above).

(N2) If (2.60) is totally linearly degenerate, i.e. in the case of Chaplygin gases $p(\rho) = A - \frac{B}{\rho}$ where A, B are positive constants, the solution for small perturbed initial data (around $(1, 0)$) exists globally in time. Roughly speaking, if a nonlinear hyperbolic conservation law system is totally linearly degenerate, then the j-characteristics

$$\partial_t s + \lambda_j(v)\partial_x s = 0$$

do not interact strongly with each other and hence the global-in-time classical solution exists for small initial data.

In the following we discuss further Riemann invariants and more general first-order system with one space variable [21].

Riemann invariants Historically Riemann introduced the concepts of Riemann invariants to solve the Riemann problem. In the case of genuine nonlinearity (otherwise we simply take $w_j = \lambda_j$), we define the (single) 1-Riemann invariant $w_1 = w_1(v) : \mathbb{R}^2 \to \mathbb{R}$ by (see (2.82) below)

$$\nabla_v w_1 = \begin{pmatrix} \frac{c(\rho)}{\rho} \\ 1 \end{pmatrix}, \quad \text{i.e. } w_1 = u + \int^\rho \frac{c(s)}{s},$$

and the (single) 2-Riemann invariant $w_2 : \mathbb{R}^2 \to \mathbb{R}$ by

$$\nabla_v w_2 = \begin{pmatrix} \frac{-c(\rho)}{\rho} \\ 1 \end{pmatrix}, \quad \text{i.e. } w_2 = u - \int^\rho \frac{c(s)}{s},$$

such that

$$r_j \cdot \nabla w_j = 0, \quad j = 1, 2.$$

This means that in the $v = (\rho, u)$-space (or manifold), w_j remains constant along the orbit of the vector r_j. As r_1, r_2 are linearly independent,

$$r_k \text{ is parallel to } \nabla_v w_j \text{ when } j \neq k,$$

∇w_j is an eigenvector corresponding to the eigenvalue λ_k, and (w_1, w_2) can serve as coordinate system in (ρ, u)-space. Let l_1, l_2 be the corresponding left eigenvectors of the eigenvalues λ_1, λ_2, then $r_j \cdot l_k = 0$ if $j \neq k$ and hence $l_k \cdot (\nabla_v w_k)^T = 0$. If v solves (2.60), then

$$\partial_t w_j = \partial_t v \cdot \nabla_v w_j$$

[21] This is not included in the exam.

$$
\begin{aligned}
&= -a(v)\partial_x v \cdot \nabla_v w_j \\
&= -(\nabla_v w_j)^T a(v)\partial_x v \\
&= -\lambda_k \partial_x v \cdot \nabla_v w_j \\
&= -\lambda_k \partial_x w_j.
\end{aligned}
$$

That is, we diagonize the system (2.60) if we take (w_1, w_2) (instead of (ρ, u)) as unknowns

$$
\begin{cases} \partial_t w_1 + \lambda_2 \partial_x w_1 = 0, \\ \partial_t w_2 + \lambda_1 \partial_x w_2 = 0. \end{cases} \tag{2.72}
$$

For example, if the pressure law reads

$$
p(\rho) = \frac{1}{2}\rho^2
$$

such that $c(\rho) = \sqrt{\rho}$, the two Riemann invariants are (up to constants)

$$
w_1 = u + 2\sqrt{\rho}, \quad w_2 = u - 2\sqrt{\rho}.
$$

We rewrite (2.60) in terms of the two Riemann invariants

$$
\partial_t \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} \partial_x \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0, \text{ with } \lambda_2 = \frac{3w_1 + w_2}{4}, \quad \lambda_1 = \frac{w_1 + 3w_2}{4}. \tag{2.73}
$$

If $v(t,x) = \phi(s(t,x))$ is j-simple wave solution (2.64)-(2.65) of (2.60) (see Figure 1a), then j-Riemann invariant w_j remains a constant in the whole (t,x)-plane by virtue of

$$
\frac{d}{ds} w_j(\phi(s)) = \phi'(s) \cdot \nabla w_j(\phi(s)) = r_j(\phi(s)) \cdot \nabla w_j(\phi(s)) = 0,
$$

and the other Riemann invariant w_k, $k \neq j$ is constant along each j-characteristic (2.65): $\{s = s(t,x)\}$ by virtue of

$$
\partial_t w_k + \lambda_j(\phi(s))\partial_x w_k = 0.
$$

Since the slope λ_j of each the above j-characteristic is determined by the values of w_1 and w_2 (see (2.73) above), the slope λ_j of each j-characteristic is indeed a constant. This verifies that j-characteristics of j-simples waves are all straight lines (as shown in Figure 1a).

2.5.3 Appendix: General first-order system with one space variable

We discuss briefly the general first-order system of the form (2.53):

$$
\partial_t v + a(v)\partial_x v = 0, \tag{2.74}
$$

where $v = (v_1, \cdots, v_n)$ and $a(v) = (a_{i,j})_{i,j=1}^n$ is a $n \times n$ matrix. We assume that $a(0)$ has real distinct eigenvalues and a has C^∞ entries.

Linear case If a is constant matrix and hence independent of v, then (2.74) is satisfied by

$$v(t,x) = \sum_{j=1}^{n} b_j(x - \lambda_j t) r_j,$$

where r_j are eigenvectors of a with eigenvalues λ_j. The initial condition $v|_{t=0} = v_0$ is satisfied if

$$\sum_{j=1}^{n} b_j(x) r_j = v_0(x),$$

that is, b_j is the component of v_0 along r_j. In this case, we have solved completely the Cauchy problem of (2.74).
In particular, if $n = 1$ and $a > 0$ is a constant, then the Cauchy problem

$$\partial_t v + a\partial_x v = 0, \quad v|_{t=0} = v_0(x)$$

has a unique solution

$$v(t,x) = v_0(x - at).$$

That is, the initial data v_0 is transported to the right in the $(x,t)-$plane with the speed a: This is completely different from the ODE equation $\partial_t v + av = 0$ whose solution is $v(t) = e^{-at}v_0$ which decays exponentially fast at infinity.
More generally, if $n = 1$ and $a = a(t,x)$ is a bounded function, then we define the characteristics $\{X(t,y) \,|\, t, y \in \mathbb{R}\}$ (i.e. the Lagrangian coordinates), which solves the ODE

$$\frac{dX(t)}{dt} = a(t, X(t)), \quad X(0) = y.$$

The solution is constant along the characteristics

$$v(t, X(t,y)) = v_0(y). \tag{2.75}$$

Simple waves in nonlinear case The decomposition of solutions in the linear case has an analogue in the nonlinear case when a depends on v. If v is in a neighborhood of 0 such that $a(v)$ has n real distinct eigenvalues

$$\lambda_1(v) < \lambda_2(v) < \cdots < \lambda_n(v),$$

and corresponding eigenvectors $r_1(v), r_2(v), \cdots, r_n(v)$. Motivated by the linear case, let $\mathbb{R} \ni s \mapsto \phi(s) \in \mathbb{R}^n$ be a parametrization of a curve, and $v(t,x) = \phi(s(t,x))$ with $s \in C^1$. Then the equation (2.74) reads

$$(\partial_t s)\phi' + (\partial_x s)a(\phi)\phi' = 0.$$

This implies that ϕ' is an eigenvector of $a(\phi)$, say

$$a(\phi)\phi' = \lambda_j(\phi)\phi',$$

then

$$\partial_t s + \lambda_j(\phi(s))\partial_x s = 0. \tag{2.76}$$

Now, conversely, let $s \mapsto \phi(s)$ be an integral curve of the eigenvector field $r_j(u)$ such that

$$\phi'(s) = r_j(\phi(s)), \tag{2.77}$$

Let $s = s(t,x)$ satisfies (2.76), then $v(t,x) = \phi(s(t,x))$ satisfies (2.74), and we call this solution j-simple wave. Along the characteristics of jth field:

$$\partial_t X(t,y) = \lambda_j(\phi(s(t,X(t,y)))), \quad X(0,y) = y,$$

we have

$$s(t,X(t,y)) = s(0,y)$$

and hence j-simple wave $v(t,x) = \phi(s(t,x))$ is constant:

$$v(t,X(t,y)) = \phi(s(t,X(t,y))) = \phi(s(0,y)) = v_0(y),$$

which implies in particular the characteristics are straight lines:

$$\partial_t X(t,y) = \lambda_j(v_0(y)), \quad X(0,y) = y.$$

Whether the solution $s(t,x)$ of (2.76) exists (globally) is questionable, see Burgers equation for the scalar case: There exists no simple waves for decreasing initial data (2.59).

Genuinely nonlinear condition VS Totally linearly degenerate The system (2.74) is called genuinely nonlinear if

$$r_j \cdot \nabla_v \lambda_j \neq 0,$$

and we normalize r_j such that it becomes, without loss of generality,

$$r_j \cdot \nabla_v \lambda_j = 1. \tag{2.78}$$

This implies that along the integral curve $s \mapsto \phi(s)$,

$$\frac{d}{ds}\lambda_j(\phi(s)) = \phi'(s) \cdot \nabla\lambda_j(\phi(s)) = r_j(\phi(s)) \cdot \nabla\lambda_j(\phi(s)) = 1.$$

We can revisit the rarefactive wave solution (2.58): For any $v_1 \in \mathbb{R}^n$, if $\phi(s)$ is defined by the ODE (2.77) and the initial data $\phi(\lambda_j(v_1)) = v_1$ (which exists at least in a neighborhood of $\lambda_j(v_1)$), then $\lambda_j(\phi(s)) = s$. One may check that $v(t,x) = \phi(\frac{x}{t})$ is a solution:

$$\begin{aligned}\partial_t(\phi(\frac{x}{t})) + a(\phi(\frac{x}{t}))\partial_x\phi(\frac{x}{t}) &= -\frac{x}{t^2}\phi'(\frac{x}{t}) + \frac{1}{t}a(\phi(\frac{x}{t}))\phi'(\frac{x}{t}) \\ &= \big(-\frac{x}{t^2} + \frac{1}{t}\lambda_j(\phi(\frac{x}{t}))\big)r_j(\phi(\frac{x}{t})) \\ &= \big(-\frac{x}{t^2} + \frac{1}{t}\frac{x}{t}\big)r_j(\phi(\frac{x}{t})) = 0.\end{aligned}$$

Generally speaking, for the nonlinear hyperbolic conservation laws, the genuinely nonlinear condition can arise blowup of smooth solutions in finite time and corresponds to the formation of shocks, e.g. for Burgers equation

$$a(v) = v, \quad \lambda(v) = v, \quad r(v) = 1,$$

the genuinely nonlinear condition

$$r(v)\cdot\partial_v\lambda(v) = 1$$

is satisfied, and we have seen the shock wave solution (2.59).
The opposite concept is totally linearly degenerate

$$r_j \cdot \nabla\lambda_j = 0, \quad j = 1, \cdots, n. \tag{2.79}$$

The nonlinear hyperbolic conservation law (2.74) satisfying the totally linearly degenerate condition (2.79) can produce global smooth small data solutions.

Shock waves for Riemann problem Unlike the scalar case $n = 1$, where the Rankine-Hugoniot condition gives immediately the speed of the shock wave in (2.59), in the system with $n \geqslant 2$, the n Rankine-Hugoniot conditions

$$c(v_+ - v_-) = f_+(v) - f_-(v)$$

for a solution of the form

$$v(t,x) = \begin{cases} v_- & \text{if } x \leqslant ct \\ v_+ & \text{if } x > ct, \end{cases} \tag{2.80}$$

implies $n-1$ conditions on v_+ given v_-. Similarly as the argument in Subsection 2.5.2, $n+1$ unknowns (v_+, σ) should satisfy n equations

$$v_+ - v_- - \sigma r_j(v_+, v_-) = 0, \tag{2.81}$$

which implies, by virtue of $r_j \cdot \nabla_v\lambda_j = 1$,

$$\lambda_j(v_+) < c = \lambda_j(v_+, v_-) < \lambda_j(v_-), \quad \sigma < 0.$$

Theorem 2.22 holds also for general $n \geqslant 1$, up to some obvious changes.

Riemann invariant In the genuinely nonlinear case, it is convenient to define j-Riemann invariants $w : \mathbb{R}^n \to \mathbb{R}$ by

$$r_j(v) \cdot \nabla_v w(v) = 0, \quad \forall v \in \mathbb{R}^n. \tag{2.82}$$

This implies that w is a constant along the integral curve $s \mapsto \phi(s)$, since

$$\frac{d}{ds} w(\phi(s)) = \phi'(s) \cdot \nabla_v w(\phi(s)) = r_j(\phi(s)) \cdot \nabla_v w(\phi(s)) = 0.$$

There are $(n-1)$ j-Riemann invariants whose gradients are linearly independent, such that the matrix $(\nabla w_1, \cdots, \nabla w_{n-1}, \nabla \lambda_j)$ is nonsingular. For the j-simple wave solution $v(t,x) = \phi(s(t,x))$ given above, $w(\phi(s(t,x)))$ is a constant on the whole (x,t)-plane.

[03.07.2023]

[07.07.2023]

3 Navier-Stokes equations

In this section we consider the initial value problem for the classical incompressible Navier-Stokes equations (1.23)

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{3.1}$$

Here $t \geqslant 0$ denotes the time variable, $x \in \mathbb{R}^N$, $N \geqslant 2$ the space variable, $u = u(t,x) : [0,\infty) \times \mathbb{R}^N \mapsto \mathbb{R}^N$ the unknown velocity vector field and $\Pi = \Pi(t,x) : (0,\infty) \times \mathbb{R}^N \mapsto \mathbb{R}$ the unknown pressure term. Compared to the classical incompressible Euler equations (2.1), the viscosity effect, which is quantified by the viscosity term $-\Delta u$, is taken into account in the fluids. We assume that the viscosity coefficient μ is a positive constant, which we take 1 for (notational) simplicity.
We summarize the counterpart of the reformulations we have done for Euler equations (2.1) (in the case of smooth and fast decaying solutions) below:

- Pressure formular. We apply div to the u-equation to arrive at the same equation for Π as for the case of Euler equations (since the vorticity term $-\Delta u$ vanishes after applying div):

$$-\Delta \Pi = \operatorname{tr}((\nabla u)^2) = \sum_{j,k=1}^{N} (\partial_j u^k \partial_k u^j).$$

Thus

$$\nabla\Pi = \nabla\Gamma * \mathrm{tr}((\nabla u)^2) =: A(u,u) \tag{3.2}$$

can be recovered from u, see (2.30) (see also (2.29) for $N = 3$ and Lemma 2.19 for $N = 2$).

- Reformulated Navier-Stokes equations. Inspired by the pressure formular and the reformulation (2.30) of Euler equations (2.1), we introduce the (Leray) projection operator

$$P = \mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}\,. \tag{3.3}$$

It is a projection operator on the divergence-free vector fields:

$$Pu = u, \quad \text{if } \mathrm{div}\, u = 0,$$

while annihilates the vector of gradient form:

$$P\nabla\Pi = 0.$$

We have indeed applied $\mathrm{Id} - P = -\nabla(-\Delta)^{-1}\mathrm{div}$ to (3.1), which annihilates the divergence-free terms $\partial_t u$ and $-\Delta u$, to derive the pressure formular:

$$\nabla\Pi \equiv (\mathrm{Id} - P)\nabla\Pi \overset{(3.1)_1}{=} -(\mathrm{Id} - P)(u\cdot\nabla u) \equiv \nabla\Gamma * \mathrm{tr}((\nabla u)^2).$$

In general we can always (formally) decompose a vector field $v : \mathbb{R}^N \to \mathbb{R}^N$ into a divergence free part Pv and a vector of gradient form $(\mathrm{Id} - P)v$ (which is called Helmholtz-decomposition):

$$\begin{aligned} &v = Pv + (\mathrm{Id} - P)v, \\ &\quad \text{with } \mathrm{div}\, Pv = \mathrm{div}\, v + (\mathrm{div}\,\nabla)(-\Delta)^{-1}\mathrm{div}\, v = \mathrm{div}\, v - \mathrm{div}\, v = 0, \\ &\quad \text{and } (\mathrm{Id} - P)v = \nabla\phi, \quad \phi := \big(-(-\Delta)^{-1}\mathrm{div}\, v\big) = -\Gamma * (\mathrm{div}\, v). \end{aligned}$$

We now apply P to $(3.1)_1$ to annihilate the pressure term, and arrive at the modified Navier-Stokes equations for u:

$$\begin{cases} \partial_t u - \Delta u = Q(u,u), \\ u|_{t=0} = u_0, \end{cases} \tag{3.4}$$

where

$$Q(u,u) = -P(u\cdot\nabla u).$$

We have shown that if $(u, \nabla\Pi)$ is a regular solution of (3.1) with divergence-free initial data u_0, then u satisfies (3.4). One can show that the converse is true, following the proof ideas for Corollary 2.18. **(Exercise: Notice the fact $\mathrm{div}\, P = 0$)**

- Another reformulation related to the convection term $u\cdot\nabla u = \sum_{k=1}^N u^k \partial_{x_k} u^j$ was not emphasized in the study of Euler equations, that is, due to $\operatorname{div} u = 0$,

$$\operatorname{div}(u \otimes u) = \sum_{k=1}^N \partial_{x_k}(u^j u^k) = \sum_{k=1}^N (u^j \partial_k u^k + \partial_k u^j u^k) = (u \cdot \nabla u).$$

  This reformulation of $u \cdot \nabla u$ in the same spirit of the conservation law-reformulation (2.55) for the Burgers' equation (2.54), such that weak solutions are well-defined in (2.56). Below we define analogously weak solutions for Navier-Stokes equations (3.1), where $u \cdot \nabla u$ is always understood as $\operatorname{div}(u \otimes u)$. Remark that in Theorem 2.17 strong solutions for Euler equations are considered, where $u \in C_x^{1,\alpha}$ is regular enough, such that $u \cdot \nabla u$ is well-defined without resorting to the reformulation $\operatorname{div}(u \otimes u)$.

- Vorticity formulation.
  - Case $N = 3$. Let $\omega = \operatorname{curl}(u) = \begin{pmatrix} \partial_{x_2} u^3 - \partial_{x_3} u^2 \\ \partial_{x_3} u^1 - \partial_{x_1} u^3 \\ \partial_{x_1} u^2 - \partial_{x_2} u^1 \end{pmatrix}$. If $(u, \nabla\Pi)$ satisfies (3.1), then ω satisfies

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla u, \tag{3.5}$$

    where u is represented by ω in terms of the Biot-Savart's in Theorem 2.9.
  - Case $N = 2$. Let $\omega = \partial_1 u^2 - \partial_2 u^1 \in \mathbb{R}$. If $(u, \nabla\Pi)$ satisfies (3.1), then ω satisfies

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = 0, \tag{3.6}$$

    where u is represented by ω in terms of the two-dimensional Biot-Savart's law in Lemma 2.19.

The following are two interesting observations which inspire the rigorous mathematical study of weak resp. strong solutions later. We assume below regular solutions, say $u \in C^1([0,\infty); \mathcal{S}(\mathbb{R}^N;\mathbb{R}^N))$, $\nabla\Pi \in C((0,\infty), \mathcal{S}(\mathbb{R}^N;\mathbb{R}^N))$.

Energy (in)equality Let us take $L^2(\mathbb{R}^N)$ inner product between the equation (3.1) and u itself, and we calculate the resulting terms one by one:

- $\int_{\mathbb{R}^N} \partial_t u \cdot u = \int_{\mathbb{R}^N} \frac{1}{2}\partial_t(|u|^2) = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |u|^2 = \frac{1}{2}\frac{d}{dt}\|u\|_{L^2(\mathbb{R}^N)}^2$,

- $\int_{\mathbb{R}^N} u\cdot\nabla u\cdot u = \int_{\mathbb{R}^N} \frac{1}{2}u\cdot\nabla(|u|^2) = -\int_{\mathbb{R}^N}\frac{1}{2}(\operatorname{div} u)|u|^2 = 0$,
- $\int_{\mathbb{R}^N} -\Delta u\cdot u = \int_{\mathbb{R}^N} |\nabla u|^2 = \|\nabla u\|^2_{L^2(\mathbb{R}^N)}$,
- $\int_{\mathbb{R}^N} \nabla\Pi\cdot u = -\int_{\mathbb{R}^N}\Pi\cdot\operatorname{div} u = 0$[22].

Thus we arrive at for all $t>0$,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|u|^2+\int_{\mathbb{R}^N}|\nabla u|^2=0,\ \text{i.e.}\ \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^N)}+\|\nabla u(t)\|^2_{L^2(\mathbb{R}^N)}=0,$$

which implies immediately the energy equality by integration in time:

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^N)}+\int_0^t\|\nabla u(t')\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}t' = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^N)}. \tag{3.7}$$

This means that we have *a priori* estimates for the solutions with the following finite time-space norms on the lefthand side:

$$\frac{1}{2}\|u\|^2_{L^\infty([0,\infty);L^2(\mathbb{R}^N))}+\|\nabla u\|^2_{L^2([0,\infty);L^2(\mathbb{R}^N))}\leqslant\frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^N)}. \tag{3.8}$$

As seen in the proofs of local- resp. global-in-time existence results of solutions to Euler equations, the (uniform) estimates (2.39) resp. (2.49) have played an essential role. The above energy inequality (3.8) for all the positive times leads then to the global-in-time existence of the *weak solutions* to (3.1), see Subsection 3.1 below. The associated topology is however not strong enough to ensure the uniqueness of weak solutions in dimension three.

A heat equation If one ignors the nonlinear convection term $u\cdot\nabla u$ in the Navier-Stokes equations, then (3.1) become

$$\begin{cases}\partial_t u-\Delta u+\nabla\Pi=0,\\ \operatorname{div} u=0,\\ u|_{t=0}=u_0.\end{cases} \tag{3.9}$$

We apply P to $(3.9)_1$ to get a heat equation for u:

$$\partial_t u-\Delta u=0,\quad u|_{t=0}=u_0. \tag{3.10}$$

Remark that since $\operatorname{div} u$ also satisfies the heat equation, u remains divergence-free if initially $\operatorname{div} u_0=0$. Thus any solution to heat equation with divergence-free initial data, together with vanishing pressure term, is also a solution of (3.9).

[22] That is, a regular divergence-free vector u and a regular vector of gradient form $\nabla\Pi$ is orthogonal in $L^2(\mathbb{R}^N;\mathbb{R}^N)$.

As explained in the lecture "Classical methods to PDEs", the Cauchy problem for the heat equation (3.10) has a unique solution **(Exercise. Verify this.)**

$$u(t,x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|\cdot|^2}{4t}} * u_0 = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_0(y)\,\mathrm{d}y\,, \quad t \geqslant 0. \quad (3.11)$$

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We introduce the following time-dependent nonnegative quantities:

$$V_m(t) = \sup{}_{|\alpha|=m} \|D_x^\alpha u(t,x)\|_{L_x^\infty(\mathbb{R}^N)}, \quad m \in \mathbb{N}_0,$$
$$W_m(t) = \sup{}_{|\alpha|=m} \|D_x^\alpha u(t,x)\|_{L_x^2(\mathbb{R}^N)}, \quad m \in \mathbb{N}_0\,.$$

The energy equality (3.7) also holds for regular solutions for the heat equation (3.10), and hence

$$W_0(t), \|W_1\|_{L^2([0,t])} \leqslant W_0(0), \quad \forall t \geqslant 0.$$

We can simply take x-derivatives to the *linear* heat equation to derive the heat equations for $D_x^\alpha u$, such that

$$W_m(t), \|W_{m+1}\|_{L^2([0,t])} \leqslant W_m(0), \quad \forall t \geqslant 0, \quad \forall m \in \mathbb{N}_0\,.$$

Indeed we have smooth solutions immediately away from the initial time, and high-order x-derivatives decay faster w.r.t. the time. To see this, by use of the explicit formula (3.11) and Young's and Hölder inequalities, one can show straightforwardly the following (decay) estimates for $t > 0$ and initial data $u_0 \in L^2$ **(Exercise. Verify this.)**

$$\begin{aligned} V_m(t) &\leqslant C_m W_0(0) t^{-\frac{2m+N}{4}}, \\ W_m(t) &\leqslant C_m W_0(0) t^{-\frac{m}{2}}, \end{aligned} \quad (3.12)$$

where C_m are some constants depending on $m \in \mathbb{N}_0$.
The question related to the solvability of (3.1) reduces then to whether the "linear" part $\partial_t u - \Delta u$ could control the "nonlinear" part $u \cdot \nabla u$ (the pressure term $\nabla \Pi$ can be recovered from $\nabla(-\Delta)^{-1}\mathrm{div}\,(u \cdot \nabla u)$ as in (3.2)). We will see that it is indeed this case if some smallness assumption either on the existence time or on the initial data is assumed, in Subsection 3.2 below.

3.1 Leray-Hopf's weak solutions

Thanks to the energy estimate (3.8), J. Leray proved the global-in-time existence of weak solutions to (3.1) in 1933 in his thesis. The weak solutions become strong solutions in dimension two, as observed by O. Ladyzhenskaya in 1959.

3.1.1 A dip on Fourier analysis and Sobolev spaces H^k

We don't follow the original proof by J. Leray [3] to show the existence results, and adopt a "modern" proof idea by use of Fourier transform instead. It is always convenient to deal with functions/problems defined in the whole space by use of Fourier transform. We recall briefly here the definition and basic properties of Fourier transform[23].

Definitions For an integrable function $f \in L^1(\mathbb{R}^N)$ defined in the whole space $\mathbb{R}^N$, we define its Fourier transform as a function $\hat{f} \in L^\infty(\mathbb{R}^N)$ below

$$\hat{f}(\xi) := \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x\,, \quad \forall \xi \in \mathbb{R}^N\,.$$

Let $\mathcal{S}(\mathbb{R}^N)$ denote the Schwartz space which consists of smooth functions which decay fast at infinity

$$\mathcal{S}(\mathbb{R}^N) = \{f \in C^\infty(\mathbb{R}^N) \,|\, \sup{}_{x\in\mathbb{R}^d,\,|\alpha|\leqslant k}(1+|x|^k)|\partial^\alpha f(x)| < \infty,\ \forall k \in \mathbb{N}\}.$$

Let $\mathcal{S}'(\mathbb{R}^N)$ denote the tempered distribution space as the dual space of Schwartz space. Since $\mathcal{S}(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$, we can define the Fourier transform of a Schwartz function $f \in \mathcal{S}(\mathbb{R}^N)$ as above, and one can check that $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^N)$ is also a Schwartz function. By duality we can extend the definition of Fourier transform to $\mathcal{S}'(\mathbb{R}^N)$ as follows

$$\langle \mathcal{F}(T), f\rangle_{\mathcal{S}',\mathcal{S}} = \langle T, \mathcal{F}(f)\rangle_{\mathcal{S}',\mathcal{S}}, \quad \forall f \in \mathcal{S}(\mathbb{R}^N),$$

such that the Fourier transform of a tempered distribution is also a tempered distribution. A typical Schwartz function is the Gaussian function $e^{-\frac{1}{2}|x|^2}$, and one can calculate that its Fourier transform is also the Gaussian function $\mathcal{F}(e^{-\frac{1}{2}|x|^2}) = e^{-\frac{1}{2}|\xi|^2}$. A typical tempered distribution is the Dirac function δ, and its Fourier transform is the constant function $\frac{1}{(2\pi)^{\frac{N}{2}}}$, since

$$\langle \mathcal{F}(\delta), f\rangle_{\mathcal{S}',\mathcal{S}} = \langle \delta, \mathcal{F}(f)\rangle_{\mathcal{S}',\mathcal{S}} = \mathcal{F}(f)|_{\xi=0} = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} f(x)\,\mathrm{d}x\; = \langle \frac{1}{(2\pi)^{\frac{N}{2}}}, f\rangle_{\mathcal{S}',\mathcal{S}}.$$

A big benefit to apply Fourier transform to PDEs is that it transforms differentiation operator to multiplication operator, and more precisely, it holds

$$\mathcal{F}(\frac{1}{i}\partial_{x_j} f)(\xi) = \xi_j \mathcal{F}(f)(\xi), \quad j = 1, \cdots, N,$$

[23] See e.g. Chapter 2, my notes for more detailed introduction to Fourier transform.

and hence more generally for multiindex $\mathcal{F}(\frac{1}{i^{|\alpha|}}D^\alpha f)(\xi) = \xi^\alpha \mathcal{F}(f)(\xi)$,

which can be checked first for $f \in \mathcal{S}(\mathbb{R}^N)$ and then for $f \in \mathcal{S}'(\mathbb{R}^N)$ by duality. Now let $f \in L^2 \subset \mathcal{S}'$, then the Plancherel's identity holds

$$\|f\|_{L^2} = \|\mathcal{F}(f)\|_{L^2}.$$

Recall the Sobolev space $H^k(\mathbb{R}^N)$ defined by

$$H^k(\mathbb{R}^N) := \{f \in L^2(\mathbb{R}^N) \mid \|f\|_{H^k} := \big(\sum_{|\alpha|\leqslant k} \|D^\alpha f\|_{L^2}^2\big)^{\frac{1}{2}} < \infty\}.$$

Hence equivalently, in terms of Fourier transform, $H^k(\mathbb{R}^N)$ can be defined by

$$H^k(\mathbb{R}^N) = \{f \in L^2(\mathbb{R}^N) \mid \|f\|_{H^k} := \big(\sum_{|\alpha|\leqslant k} \|\xi^\alpha \mathcal{F}(f)(\xi)\|_{L^2}^2\big)^{\frac{1}{2}} < \infty\}.$$

L^2-Functions with compactly supported Fourier transform If $f \in L^2(\mathbb{R}^N)$ and $\mathcal{F}(f)$ has compact support, say $\operatorname{supp}\mathcal{F}(f)$ is included in a ball $B_n(0)$ with radius n, then f is indeed smooth: $f \in H^k(\mathbb{R}^N)$, $\forall k \in \mathbb{N}$. Indeed, we simply calculate

$$\begin{aligned}\|f\|_{H^k} &= \Big(\sum_{|\alpha|\leqslant k} \|\xi^\alpha \mathcal{F}(f)(\xi)\|_{L^2}^2\Big)^{\frac{1}{2}} \\ &\leqslant \Big((\sum_{|\alpha|\leqslant k} 1) n^{2k} \|\mathcal{F}(f)\|_{L^2}^2\Big)^{\frac{1}{2}} \leqslant C_k n^k \|f\|_{L^2}, \quad \forall k \in \mathbb{N}.\end{aligned}$$

Motivated by this analysis, we introduce the low-frequency cut-off operator

$$\mathcal{P}_n := 1_{B_n}(D), \text{ i.e. } \mathcal{F}(\mathcal{P}_n f) = \mathcal{F}(f)|_{\{|\xi|<n\}}, \tag{3.13}$$

such that

- it is a regularizing operator on L^2 in the sense that $\mathcal{P}_n : L^2 \to H^\infty = \cap_{k\in\mathbb{N}} H^k$
- it is an approximation operator on L^2 in the sense that

$$\forall f \in L^2, \quad \|(\mathrm{Id} - \mathcal{P}_n)f\|_{L^2} = \|\hat{f}|_{(B_n)^C}\|_{L^2} \overset{n\to\infty}{\to} 0.$$

It is convenient to apply $\mathcal{P}_n$ to (the nonlinearities in) some PDE, then to construct a regular solution u_n of the regularized PDE, and finally to show the convergence of the sequence (u_n) to some limit u, which is expected to solve the original PDE. In this way one can show the existence of solutions.

3.1.2 Global-in-time existence of weak solutions

We call a divergence-free vector-valued function $u \in L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^N; \mathbb{R}^N)$ a weak solution of (3.1) if the following equality holds

$$\begin{aligned} 0 = \int_0^\infty \int_{\mathbb{R}^N} (u \cdot \partial_t \varphi + u \otimes u : \nabla \varphi + u \cdot \Delta \varphi) \, \mathrm{d}x \, dt \\ + \int_{\mathbb{R}^N} u_0(x) \cdot \varphi(0,x) \, \mathrm{d}x \,, \end{aligned} \tag{3.14}$$

for any test function $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^N; \mathbb{R}^N)$ with $\operatorname{div} \varphi = 0$. **Exercise:** Show that regular solutions of (3.1) satisfy (3.14) by integration by parts.

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The following existence result of weak solutions is due to J. Leray.

Theorem 3.1 (Existence of weak solutions of (3.1))**.** *Let $N = 2$ or 3. Let u_0 be a divergence-free vector field in $(L^2(\mathbb{R}^N))^N$. Then there exists a weak solution $u = u(t,x) \in C([0,\infty); L^2_w(\mathbb{R}^N))$ of (3.1) satisfying the energy inequality:*

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^N)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^N)} \, \mathrm{d}t' \leqslant \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^N)}. \tag{3.15}$$

Ideas of proof. The procedure of the proof is similar as in the proof of Theorem 2.17.

Step 1 Construction of a sequence of smooth solutions

$$(u_n) \subset C^1([0,\infty); (H^{N+1}(\mathbb{R}^N))^N)$$

of the regularized Cauchy problem of (3.4):

$$\partial_t v = \Delta v + \mathcal{P}_n Q(v,v), \quad v(0,x) = \mathcal{P}_n u_0(x), \tag{3.16}$$

where $\mathcal{P}_n = 1_{B_n}(D)$, $n \in \mathbb{N}$ is the low-frequency cut-off operator given in (3.13), and $Q(v,v) = -P(v \cdot \nabla v)$. Here $P = \mathrm{Id} + \nabla(-\Delta)^{-1}\mathrm{div}$ is the Leray projector (3.3), which is also a Calderon-Zygmund operator and hence satisfies the L^p-Estimates in Lemma 2.10:

$$\|Pv\|_{L^p} \leqslant C_p \|v\|_{L^p}, \quad \forall p \in (1,\infty). \tag{3.17}$$

Notice that any regular solution v, say $v \in C^1([0,\infty); H^{N+1})$, of (3.16)

- is divergence-free. Indeed, we apply div to (3.16) to arrive at the free heat equation for $\operatorname{div} v$

$$\partial_t(\operatorname{div} v) - \Delta(\operatorname{div} v) = 0, \quad (\operatorname{div} v)(0, x) = 0,$$

where we used the community between $\mathcal{P}_n$ and div, $\operatorname{div} P = 0$ and $\operatorname{div} u_0 = 0$. Hence $\operatorname{div} v = 0$ for all the times.

- has compactly supported Fourier transform such that $v = \mathcal{P}_n v$. Indeed, we apply Fourier transform to (3.16) to arrive at

$$\partial_t \hat{v}(t,\xi) + |\xi|^2 \hat{v}(t,\xi) = 1_{B_n}(\xi)\widehat{Q(v,v)}(t,\xi), \quad \hat{v}(0,\xi) = 1_{B_n}(\xi)\widehat{u_0}(\xi).$$

We view ξ as a parameter, and solve the above ordinary differential equation to get the solution

$$\hat{v}(t,\xi) = e^{-|\xi|^2 t}\big(1_{B_n}(\xi)\widehat{u_0}(\xi)\big) + \int_0^t e^{-|\xi|^2(t-t')} 1_{B_n}(\xi)\widehat{Q(v,v)}(t',\xi)dt'. \tag{3.18}$$

Fix $t > 0$, and we see from the above that

$$\operatorname{Supp}(\hat{v}(t,\xi)) \subset B_n, \text{ that is } v = 1_{B_n}(D)v = \mathcal{P}_n v.$$

- satisfies the energy equality (3.7)

$$\frac{1}{2}\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v\|_{L^2}^2 dt' = \frac{1}{2}\|\mathcal{P}_n u_0\|_{L^2}^2. \tag{3.19}$$

Indeed, as in the derivation of (3.7), we take L^2-inner product between (3.16) with v itself. Then (3.19) follows, by virtue of the fact

$$\langle f, \mathcal{P}_n g\rangle_{L^2} = \int_{\mathbb{R}^N} \hat{f}(\xi) 1_{B_n}(\xi)\bar{\hat{g}}(\xi) = \langle \mathcal{P}_n f, g\rangle_{L^2},$$

and $\mathcal{P}_n v = v$, $Pv = v$, $\operatorname{div} v = 0$:

$$\langle v, \mathcal{P}_n Q(v,v)\rangle_{L^2} = \langle v, -(v\cdot\nabla v)\rangle_{L^2} = 0.$$

- satisfies (3.14) in the following sense:

$$\begin{aligned} 0 = &\int_0^\infty \int_{\mathbb{R}^N} (v\cdot\partial_t\varphi + \mathcal{P}_n(v\otimes v) : \nabla\varphi + v\cdot\Delta\varphi)\,\mathrm{d}x\,dt \\ &+ \int_{\mathbb{R}^N} \mathcal{P}_n u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x\,, \end{aligned} \tag{3.20}$$

for any test function $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R}^N;\mathbb{R}^N)$ with $\operatorname{div}\varphi = 0$. Indeed, it follows as for (3.14).

We claim that (3.16) has a unique regular solution in $C^1([0,\infty); H^{N+1})$. Indeed, (3.16) is an ordinary differential equation in the following subspace of H^{N+1}

$$H_n^{N+1} := \{v(x) \in H^{N+1} \,|\, \mathrm{Supp}\,(\hat{v}) \subset B_n\},$$

where the righthand side $\Delta v + \mathcal{P}_n Q(v,v)$ has uniformly-in-time bounded Lipschitz constant, since **(Exercise)**

- by use of Sobolev embedding $H^N(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$, Hölder's inequality $\|v\cdot\nabla v\|_{L^2} \leqslant \|v\|_{L^2}\|\nabla v\|_{L^\infty} \leqslant \|v\|_{L^2}\|\nabla v\|_{H^N}$ and the L^2-estimate of the projector P, the righthand side of (3.16) is Lipschitz continuous in H_n^{N+1} such that

$$\begin{aligned}\|\Delta v + \mathcal{P}_n Q(v,v)\|_{H^{N+1}} &\leqslant C_{N+1} n^{N+1} \|\Delta v + \mathcal{P}_n Q(v,v)\|_{L^2}\\ &\leqslant C_{N+1} n^{N+1} (\|v\|_{H^N} + \|v\|_{L^2}\|v\|_{H^{N+1}})\\ &\leqslant C_{N+1} n^{N+1} (\|v\|_{H^N} + \|v\|^2_{H^{N+1}}),\end{aligned}$$

- and hence Cauchy-Lipschitz theorem implies a local-in-time unique solution
$$u_n \in C^1([0,T]; H_n^{N+1})$$
for some $T \in (0,\infty)$,

- and (similarly as in the proof of Theorem 2.21) we can indeed take $T = \infty$ by virtue of the uniform Lipschitz constant of the righthand side which comes from (3.19)

$$\|u_n\|_{L^\infty([0,\infty);L^2)} \leqslant \|\mathcal{P}_n u_0\|_{L^2} \leqslant \|u_0\|_{L^2} < \infty.$$

Step 2 Convergence by uniform bounds and compactness. Since u_n satisfies (3.19), the following uniform estimate holds:

$$\frac{1}{2}\|u_n\|^2_{L^\infty([0,\infty);L^2)} + \|\nabla u_n\|^2_{L^2([0,\infty);L^2)} \leqslant \frac{1}{2}\|u_0\|^2_{L^2}, \tag{3.21}$$

and hence by interpolation inequality for $N = 2, 3$[24]

$$\|u_n\|_{L^{\frac{8}{N}}([0,\infty);L^4)} \leqslant C\|u_n\|_{L^\infty([0,\infty);L^2)}^{\frac{4-N}{4}}\|\nabla u_n\|_{L^2([0,\infty);L^2)}^{\frac{N}{4}} \leqslant C\|u_0\|_{L^2}.$$

This implies

$$\begin{aligned}
&\|\Delta u_n\|_{L^2([0,\infty);H^{-1})} \leqslant \|\nabla u_n\|_{L^2([0,\infty);L^2)} \leqslant \|u_0\|_{L^2},\\
&\|\mathcal{P}_n Q(u_n, u_n)\|_{L^{\frac{4}{N}}([0,\infty);H^{-1})} \leqslant C\|u_n\|^2_{L^{\frac{8}{N}}([0,\infty);L^4)} \leqslant C\|u_0\|^2_{L^2},
\end{aligned}$$

and hence the uniform bound for $\partial_t u_n$ on any fixed finite time interval

$$\|\partial_t u_n\|_{L^{\frac{4}{N}}([0,T];H^{-1})} \leqslant C(T, \|u_0\|_{L^2}). \tag{3.22}$$

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This gives compactness of the sequence (u_n) w.r.t. the time variable since $\frac{4}{N} > 1$, and the uniform bound (3.21) implies the compactness w.r.t. the x-variable *locally*. More precisely, we state here the celebrated Aubin-Lions' Lemma without proof [25]:

Lemma 3.2 (Aubin-Lions' Lemma)**.** *Let X_0, X_1 be separable and reflexive Banach spaces, and X be a Banach space such that $X_0 \subset X \subset X_1$ continuously and the embedding $X_0 \subset X$ is compact. Then the embedding*

$$\Big\{u \in L^p([0,T];X_0) \,|\, \partial_t u \in L^q([0,T];X_1)\Big\} \subset L^p([0,T];X), \quad T \in (0,\infty)$$

is compact if $p \in [1,\infty)$ and $q \in [1,\infty]$. If $p = \infty$, $q > 1$, then the subset is compactly embedded in $C([0,T];X)$.

[24] This is the celebrated Gagliardo-Nirenberg's inequality. See e.g. Proposition 2.2 & Proposition 2.3, my notes for the proof of interpolation/embedding inequalities and the relationship between Besov spaces and Lebesgue spaces (by use of Fourier analysis) respectively:

$$\begin{aligned}
\|u\|_{L^4(\mathbb{R}^N)} \leqslant C\|u\|_{\dot{B}^0_{4,1}(\mathbb{R}^N)} &\leqslant C\|u\|^{\frac{4-N}{4}}_{\dot{B}^{-\frac{N}{4}}_{4,\infty}(\mathbb{R}^N)}\|u\|^{\frac{N}{4}}_{\dot{B}^{1-\frac{N}{4}}_{4,\infty}(\mathbb{R}^N)}\\
&\leqslant C\|u\|^{\frac{4-N}{4}}_{\dot{B}^0_{2,\infty}(\mathbb{R}^N)}\|u\|^{\frac{N}{4}}_{\dot{B}^1_{2,\infty}(\mathbb{R}^N)} \leqslant C\|u\|^{\frac{4-N}{4}}_{L^2(\mathbb{R}^N)}\|\nabla u\|^{\frac{N}{4}}_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

[25] The proof can be found in Section 1.5 of J.L. Lions' book "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires", 1969, Dunod, Paris. More general cases can be found in T. Roubíček's article "A generalization of the Lions-Temam compact imbedding theorem, Casopis pest. mat. 115, 338-342.

For any fixed ball B_k, we take $X_0 = H^1(B_k)$, $X = L^2(B_k)$, $X_1 = H^{-1}(B_k)$ (which is the dual space of $H_0^1(B_k) := \overline{C_c^\infty(B_k)}^{\|\|_{H^1}}$), $p = 2$, $q = \frac{4}{N}$, then the sequence $\{u_n\}$ is compact in $L^2([0,T]; L^2(B_k))$. By Cauchy's diagonalization argument, there exists a subsequence (u_{n_k}) converges to some (weak) limit such that

$$u_{n_k} \to u \text{ in } L^2([0,T]; L^2(B_{k_0})), \quad \forall k_0 \in \mathbb{N}\,.$$

The limit u satisfies (3.15) by applying Fatou's lemma to (3.21). Finally, since (u_{n_k}) satisfies (3.20), the limit u satisfies (3.14), and hence is a global-in-time weak solution of (3.1) since $\operatorname{div} u = 0$. Since $u \in L^\infty([0,\infty); L^2)$ and $\partial_t u \in L_{\mathrm{loc}}^{\frac{4}{N}}([0,\infty); H^{-1})$, the solution is continuous in time w.r.t. the weak topology of $L^2(\mathbb{R}^N; \mathbb{R}^N)$: $u \in C([0,\infty); L^2_w)$. □

Remark 3.3. *In three dimensional case, the energy equality* (3.7)*, the continuity in time w.r.t.* $L^2(\mathbb{R}^N;\mathbb{R}^N)$*-strong topology or the uniqueness result does not necessarily hold for weak solutions.*

3.1.3 Two-dimensional case

Theorem 3.4 (Energy Equality & Uniqueness & Continuity of weak solutions in dimension two)**.** *Let $N = 2$. Then the weak solution given in Theorem 3.1 is unique, continuous (strongly) in $L^2(\mathbb{R}^2)$ and satisfies the energy equality:*

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^2)}\,\mathrm{d}t' = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^2)}, \quad \forall t > 0. \tag{3.23}$$

Proof. The equality (3.14) implies that the equation

$$\partial_t u - \Delta u = -P\mathrm{div}\,(u \otimes u)$$

holds in the distribution sense. By the interpolation inequality used in Step 2 in the proof of Theorem 3.1, the weak solutions u satisfying (3.15) belongs to $L^4([0,\infty); (L^4(\mathbb{R}^2))^2)$, and hence the above equation holds in $L^2([0,\infty); (H^{-1}(\mathbb{R}^2))^2)$.
For any fixed $T > 0$, the weak solutions $u \in L^2([0,T]; (H^1(\mathbb{R}^2))^2)$ stays in the dual space of $L^2([0,T]; (H^{-1}(\mathbb{R}^2))^2)$. Hence we can test *rigorously* the equation above by u itself to derive (3.23), since the calculation to derive (3.7) still holds:

- $\int_0^T \langle \partial_t u, u\rangle_{H^{-1},H^1} dt = \frac{1}{2}\int_0^T \frac{d}{dt}\|u\|^2_{L^2} dt = \frac{1}{2}\|u(T)\|^2_{L^2} - \frac{1}{2}\|u_0\|^2_{L^2}$,

- $\langle -\Delta u, u\rangle_{L^2_T H^{-1}, L^2_T H^1} = \|\nabla u\|^2_{L^2_T L^2}$,
- $\langle -P\mathrm{div}\,(u\otimes u), u\rangle_{L^2_T H^{-1}, L^2_T H^1} = \langle u\otimes u, \nabla u\rangle_{L^2_T L^2, L^2_T L^2} = \frac{1}{2}\langle u, \nabla |u|^2\rangle_{L^2_T H^1, L^2_T H^{-1}} = -\frac{1}{2}\langle \mathrm{div}\, u, |u|^2\rangle_{L^2_T L^2, L^2_T L^2} = 0$.

By the time continuity of L^2-norm given by (3.23), the weak solution $u \in C([0,\infty); L^2_w)$ given in Theorem 3.1 is indeed continuous in L^2-strong topology.
The uniqueness of the weak solutions satisfying (3.15) follows similarly. Let u, v be two such weak solutions with the same initial data. Then their difference $w \in L^\infty([0,\infty); L^2)$ with $\nabla w \in L^2([0,\infty); L^2)$ satisfies

$$\partial_t w - \Delta w = -P\mathrm{div}\,(w \otimes u + v \otimes w)$$

in $L^2([0,\infty); H^{-1})$. We can test it by w itself, and derive similar energy equality for w, up to a correction term due to $-P\mathrm{div}\,(w\otimes u)$: Young's inequality and Gronwall's inequality finally imply $w = 0$ all the time **(Exercise)**. □

3.2 Kato's strong solutions in space dimension three

Scaling property We observe the following scaling invariance property of the Navier-Stokes equations (3.1): If $(u,\Pi)(t,x)$ is a solution of (3.1) with the initial data u_0 on the time interval $[0,T]$, then the rescaled pair

$$(u_\lambda, \Pi_\lambda)(t,x) = (\lambda u, \lambda^2 \Pi)(\lambda^2 t, \lambda x), \quad \lambda > 0, \tag{3.24}$$

is a solution of (3.1) of the initial data $u_{0,\lambda}(x) = \lambda u_0(\lambda x)$ on the time interval $[0, \lambda^{-2}T]$. We calculate the $L^p(\mathbb{R}^N)$-norm of $u_{0,\lambda}$:

$$\|u_{0,\lambda}\|_{L^p(\mathbb{R}^N)} = \lambda^{1-\frac{N}{p}} \|u_0\|_{L^p(\mathbb{R}^N)}.$$

Heuristically, we then divide the exponent p of the Lebesgue space L^p into three cases:

- $p > N$ (subcritical case)
  As $\lambda \to 0$, $\|u_{0,\lambda}\|_{L^p(\mathbb{R}^N)} \to 0$ and the rescaled solution u_λ exists on the time interval $[0, \lambda^{-2}T]$ with $\lambda^{-2}T \to \infty$. This is the most favourable situation in well-posedness issue: we can make both the small initial norm and the long time interval at the same time.

- $p = N$ (critical case)
  It is easy to see that the $L^p(\mathbb{R}^N)$-norm is invariant under the scaling: $\|u_{0,\lambda}\|_{L^N(\mathbb{R}^N)} = \|u_0\|_{L^N(\mathbb{R}^N)}$, and as $\lambda \to 0$ the rescaled existing time interval is still $[0, \lambda^{-2}T]$ with $\lambda^{-2}T \to \infty$. This is always a unclear situation.

- $p < N$ (supercritical case)
  In this case as $\lambda \to 0$, $\|u_{0,\lambda}\|_{L^p(\mathbb{R}^N)} \to \infty$ as $\lambda^{-2}T \to \infty$, that is, the growing norm corresponds to longer time interval. Blowup may happen in this situation.

In the two dimensional case $N = 2$, we have established the global-in-time well-posedness results for the Navier-Stokes equations in Theorem 3.4 in the critical case with $u_0 \in (L^2(\mathbb{R}^2))^2$, which happens to be the energy space, where the energy estimates (3.23) are at hand a priori.
While in the three dimensional case $N = 3$, the weak solutions are indeed in the supercritical case, and we do not expect the uniqueness results for the weak solutions with $u_0 \in (L^2(\mathbb{R}^3))^3$. We are going to consider the critical case $u_0 \in (L^3(\mathbb{R}^3))^3$ in dimension three.

[24.07.2023]

[28.07.2023]

Reformulation of NS by use of Fourier transform Recall the definition of the Leray projector (3.3): $P = \text{Id} + \nabla(-\Delta)^{-1}\text{div}$, and we have applied the operator P to the equation (3.1) to arrive at (3.4):

$$\begin{cases} \partial_t u - \Delta u = Q(u,u), \\ u|_{t=0} = u_0, \end{cases} \tag{3.25}$$

where $Q(u,u) = -P\text{div}\,(u \otimes u)$.
Recall the definition of Fourier transform in Subsection 3.1.1, and one calculates

$$\widehat{(Pv)}^j(\xi) = \widehat{v^j} - \sum_{k=1}^{d} \frac{\xi_j \xi_k}{|\xi|^2}\widehat{v^k} = \sum_{k=1}^{d}(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})\widehat{v^k}, \tag{3.26}$$

and

$$\widehat{Q(u,u)}^j = -P\widehat{\text{div}\,(u\otimes u)}^j = -\sum_{k,l=1}^{d}(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})(i\xi_l)\widehat{u^k u^l}. \tag{3.27}$$

We apply Fourier transform to (3.25) to derive

$$\partial_t \hat{u} + |\xi|^2 \hat{u} = \widehat{Q(u,u)}, \quad \hat{u}(0) = \widehat{u_0}(\xi),$$

and as (3.18) we arrive at the following Duhamel's formular

$$\widehat{u^j}(t,\xi) = e^{-t|\xi|^2}\widehat{u_0}^j(\xi) - \sum_{k,l=1}^{d}\int_0^t e^{-(t-t')|\xi|^2}(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})(i\xi_l)u^k\widehat{(t')u^l}(t')\,\mathrm{d}t'\,.$$

Denote

$$e^{t\Delta}u_0 = \mathcal{F}^{-1}(e^{-t|\xi|^2}\widehat{u_0}^j(\xi)),$$
$$\Gamma^j_{kl}(t,\cdot) = (2\pi)^{-\frac{3}{2}}\mathcal{F}^{-1}\Big(-e^{-t|\xi|^2}(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2})(i\xi_l)\Big),$$

then (3.25) is reformulated as follows[26]

$$u(t,x) = e^{t\Delta}u_0 + \sum_{k,l=1}^{d}\int_0^t \Gamma_{kl}(t-t',\cdot) * (u^k u^l)(t',\cdot)\,\mathrm{d}t' \,, \tag{3.28}$$

that is, we search for the fixed point of the map

$$u \mapsto (e^{t\Delta}u_0) + B(u,u), \quad B(u,u) := \int_0^t \Gamma_{kl}(t-t',\cdot) * (u^k u^l)(t',\cdot)\,\mathrm{d}t' \,. \tag{3.29}$$

Heat equation revisited We have shown the decay estimates (3.12) for the solution[27]

$$u(t,x) = e^{t\Delta}u_0 = \mathcal{F}^{-1}(e^{-t|\xi|^2}\widehat{u_0}^j(\xi)) = (4\pi t)^{-\frac{N}{2}}e^{-\frac{|\cdot|^2}{4t}} * u_0$$

of the Cauchy problem for the heat equation (3.10) with L^2-initial data:

$$\partial_t u - \Delta u = 0, \quad u|_{t=0} = u_0.$$

Now $N = 3$ and we take $u_0 \in (L^3(\mathbb{R}^3))^3$. Similar as the derivation of (3.12), we apply Young's inequality to derive for $\beta \geqslant 3$ **(Exercise)**

$$\|e^{t\Delta}u_0\|_{L^\beta(\mathbb{R}^3)} \leqslant Ct^{-\frac{1}{2}(1-\frac{3}{\beta})}\|u_0\|_{L^3(\mathbb{R}^3)}.$$

For any $p \in [1,\infty]$, $T \in (0,\infty)$, we define Kato's space

$$K_p(T) = \{u \in C((0,T];(L^p(\mathbb{R}^3))^3) \mid \|u\|_{K_p(T)} := \sup_{t\in(0,T]} t^{\frac{1}{2}(1-\frac{3}{p})}\|u(t)\|_{L^p(\mathbb{R}^3)} < \infty\}, \tag{3.30}$$

then

$$e^{t\Delta}u_0 \in K_\beta(T) \text{ and } \|e^{t\Delta}u_0\|_{K_\beta(T)} \leqslant C\|u_0\|_{L^3(\mathbb{R}^3)}, \quad \forall \beta \geqslant 3. \tag{3.31}$$

We aim to find fixed point of (3.29) in $K_6(T)$, and this requires the study of the function Γ_{kl}.
We can show the local-in-time well-posedness result of (3.25) in different functional frameworks and here we will follow Kato's L^p approach to show the well-posedness result of (3.25) in $L^3(\mathbb{R}^3)$ in three dimensional case.

[26] We have used the fact that $\mathcal{F}(f * g) = (2\pi)^{\frac{N}{2}}\mathcal{F}(f)\mathcal{F}(g)$.

[27] We notice that $\mathcal{F}^{-1}(e^{-t|\xi|^2}) = (2t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$, $t > 0$.

Theorem 3.5. *Let $u_0 \in (L^3(\mathbb{R}^3))^3$. Then there exists a positive time T and a unique solution $u \in C([0,T];(L^3(\mathbb{R}^3))^3)$ of the initial value problem (3.25). There exists a positive constant c such that if $\|u_0\|_{L^3} \leqslant c$ then T can be chosen as $+\infty$.*

Proof. The ideas of proof are the same as before, and we sketch them below[28]. Here we have to first establish the a priori estimates, which play the same role as the energy inequality (3.15) in the proof for the existence of weak solutions.

Step 1 A priori estimate Recall the reformulation (3.28). A straightforward calculation (similar as the derivation of (3.12), which we do not do here) yields the following pointwise bound for Γ

$$|\Gamma^j_{kl}| \leqslant C\min\{|x|^{-4}, t^{-2}\} \leqslant C(|x|+\sqrt{t})^{-4}.$$

Hence by Young's inequalities w.r.t. x and t-variables we have

$$\begin{aligned}&\left\|\int_0^t \Gamma_{kl}(t-t',\cdot) * (u^k u^l)(t')\,\mathrm{d}t'\right\|_{K_\beta(T)} \leqslant C\|u\|_{K_p(T)}\|u\|_{K_q(T)},\\ &\text{if } \tfrac{1}{\beta} \leqslant \tfrac{1}{p}+\tfrac{1}{q} < \tfrac{1}{3}+\tfrac{1}{\beta},\ \tfrac{1}{p}+\tfrac{1}{q} \leqslant 1.\end{aligned} \tag{3.32}$$

To conclude, we have arrived at the following a priori estimates:

$$\begin{aligned}&\|u\|_{K_\beta(T)} \leqslant C\big(\|u_0\|_{L^3} + \|u\|_{K_p(T)}\|u\|_{K_q(T)}\big),\\ &\forall \beta \geqslant 3 \text{ s.t. } \frac{1}{\beta} \leqslant \frac{1}{p}+\frac{1}{q} < \frac{1}{3}+\frac{1}{\beta} \leqslant \frac{2}{3}, \quad \forall T>0.\end{aligned} \tag{3.33}$$

Step 2 Existence & Uniqueness of the solution in $K_6(T)$
We have established the a priori estimate (3.33) for the solution (3.28) to (3.25) in Step 1. We would like to use the contraction mapping argument to show the existence of the solution in the Banach space $K_6(T)$ under some smallness condition on the time T or on the initial data $\|u_0\|_{L^3}$. That is, we search for the fixed point of the map $u \mapsto e^{t\Delta}u_0 + B(u,u)$ given in (3.29) in $K_6(T)$.
We have shown $e^{t\Delta}u_0 \in K_6(T)$ for any $T \in (0,\infty)$ by (3.31). As $\Gamma \in C((0,\infty);(L^\alpha(\mathbb{R}^3))^9)$, $\forall \alpha \in [1,\infty)$, (3.32) implies

$$B : K_6(T)\times K_6(T) \to K_6(T), \text{ with } \|B(u,v)\|_{K_6(T)} \leqslant C\|u\|_{K_6(T)}\|v\|_{K_6(T)}.$$

1. Case of arbitrarily large initial data and small existence time.

[28] See e.g. my notes for the detailed proof of Theorem 3.4 there.

For any $u_0 \in L^3(\mathbb{R}^3)$, for any $\varepsilon > 0$, there exists $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that $\|u_0 - \varphi\|_{L^3(\mathbb{R}^3)} < \varepsilon$. On the other hand, $\|e^{t\Delta}\varphi\|_{L^\infty([0,T];L^6)} \leqslant C\|\varphi\|_{L^6}$. Thus

$$\begin{aligned}\|e^{t\Delta}u_0\|_{K_6(T)} &\leqslant \|e^{t\Delta}(u_0-\varphi)\|_{K_6(T)} + \|e^{t\Delta}\varphi\|_{K_6(T)} \\ &\leqslant C\|u_0-\varphi\|_{L^3} + CT^{\frac{1}{2}(1-\frac{3}{6})}\|\varphi\|_{L^6} \leqslant C\varepsilon + CT^{\frac{1}{4}}\|\varphi\|_{L^6}.\end{aligned}$$

We can choose T sufficiently small (depending on u_0, ε) such that

$$\|e^{t\Delta}u_0\|_{K_6(T)} \leqslant C\varepsilon. \tag{3.34}$$

Therefore for $\varepsilon > 0$, $T > 0$ sufficiently small, we derive from the contraction mapping argument that there exists a unique fixed point u of the map $u \mapsto e^{t\Delta}u_0 + B(u,u)$ in the Banach space $K_6(T)$, with

$$\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}. \tag{3.35}$$

2. Case of small initial data and arbitrarily large existence time.

   If $\|u_0\|_{L^3(\mathbb{R}^3)} < c$, then

$$\|e^{t\Delta}u_0\|_{K_6(T)} \leqslant C\|u_0\|_{L^3} \leqslant Cc, \quad \forall T \in (0,\infty).$$

   Hence in the small data case that $c > 0$ is sufficiently small, there exists a unique fixed point $u \in K_6(T)$ for any $T \in (0,\infty)$, with $\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}$.

Step 3 Final check: Continuity and Uniqueness
Although we have showed in Step 2 the existence and the uniqueness of the solution $u \in K_6(T)$ such that $\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}$ is small enough, we have to prove further $u \in C([0,T];L^3)$ and the uniqueness of the solution therein.
Now $u \in K_6(T)$ with $\|u\|_{K_6(T)} \leqslant 2\|e^{t\Delta}u_0\|_{K_6(T)}$ is the known function and we would like to show

$$u = a + \tilde{u} \in C([0,T];L^3), \text{ with } a := e^{t\Delta}u_0 \text{ and } \tilde{u} := B(u,u).$$

Obviously $a = e^{t\Delta}u_0 \in C([0,T];L^3)$. As $u \in K_6(T)$, we infer from the derivation of the estimate (3.32) (with $\beta = 3$) that $\tilde{u} = B(u,u) \in C((0,T];L^3)$ and for any $t \in (0,T)$,

$$\|\tilde{u}\|_{L^\infty([0,t];L^3)} \leqslant C\|u\|^2_{K_6(t)} \leqslant 4C\|e^{t\Delta}u_0\|^2_{K_6(t)}, \tag{3.36}$$

where the righthand side tends to zero as $t \to 0^+$ (recalling the decomposition $e^{t\Delta}u_0 = e^{t\Delta}(u_0 - \varphi) + e^{t\Delta}\varphi$). This implies the continuity of $\tilde{u}$ at time zero and hence $\tilde{u} \in C([0,T]; L^3)$.
The proof of the uniqueness of the solutions in $C([0,T]; L^3)$ is more involved due to the low regularity assumption. It is more convenient to show the uniqueness in a weaker topology, say $L^\infty([0,t]; \dot{H}^{-\frac{1}{2}})$, and hence the uniqueness in the stronger topology holds.

□

Remark 3.6. *We have shown the well-posedness results for the three-dimensional Navier-Stokes equations* (3.1) *in the critical Lebesgue space* $(L^3(\mathbb{R}^3))^3$ *in the sense of* (3.25)*, or more precisely* (3.28)*. The obtained solution* $u \in C([0,T]; (L^3(\mathbb{R}^3))^3)$ *together with the pressure term given in* (3.2) *satisfies* (3.1) *uniquely.*

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