Nonlinear Boundary Value Problems:  
1st problem sheet

Exercise 1

Consider the steady flow of a viscous incompressible fluid through a pipe of length $L > 0$ and of height $1$.

Find a solution of the Navier-Stokes equations with viscous term $\eta \Delta v$ ($\eta$ a strictly positive constant) and without force $f$ of the form $v(x, y) = (u(x, y), 0)$ ($0 \leq x \leq L, 0 \leq y \leq 1$) that satisfies the boundary conditions

$$p(0, y) = p_1, \quad p(L, y) = p_2, \quad u(x, 0) = u(x, 1) = 0 \quad (p_1, p_2 \in \mathbb{R}).$$

Exercise 2: Bernoulli’s theorem

Consider Euler’s equation

$$\rho(v \cdot \nabla)v + \nabla p = 0$$

for the steady inviscid flow of a gas without external force. Moreover, let a smooth pressure-density relation $p = p(\rho)$ be given. A $C^1$-curve $\gamma : [0,1] \to \mathbb{R}^3$ is called a streamline if $\gamma'(s) = v(\gamma(s))$ for all $s \in (0,1)$. Show that there is a function $w$ such that

$$\frac{1}{2}|v|^2 + w(\rho)$$

stays constant along any streamline.

$\textbf{Hint:}$ Use $\frac{1}{2}\nabla|v|^2 = (v \cdot \nabla)v + v \times \text{curl}(v)$.
Exercise 3

Let $H_1, H_2$ be Hilbert spaces and $A : H_1 \to H_2$ a bounded linear operator. Show that $u_n \to u$ in $H_1$ implies $Au_n \to Au$ in $H_2$.

Exercise 4

Let $H$ be an infinite-dimensional Hilbert space and $(u_n)_{n \in \mathbb{N}}$ an orthonormal system. Prove that $u_n \to 0$ but $(u_n)$ contains no norm convergent subsequence.