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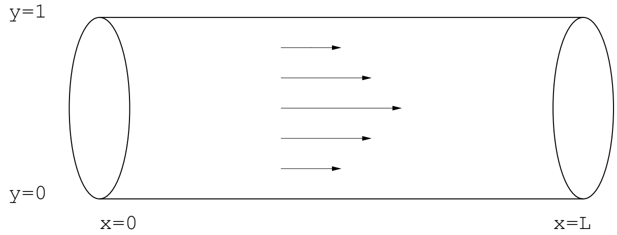
Nonlinear Boundary Value Problems:
1st problem sheet

Exercise 1

Consider the steady flow of a viscous incompressible fluid through a pipe of length $L > 0$ and of height 1.



Find a solution of the Navier-Stokes equations with viscous term $\eta\Delta v$ (η a strictly positive constant) and without force f of the form $v(x,y) = (u(x,y), 0)$ ($0 \leq x \leq L$, $0 \leq y \leq 1$) that satisfiess the boundary conditions

$$p(0,y) = p_1,\ p(L,y) = p_2, \quad u(x,0) = u(x,1) = 0 \qquad (p_1, p_2 \in \mathbb{R}).$$

Exercise 2: Bernoulli's theorem

Consider Euler's equation

$$\rho(v \cdot \nabla)v + \nabla p = 0$$

for the steady inviscid flow of a gas without external force. Moreover, let a smooth pressure-density relation $p = p(\rho)$ be given. A C^1-curve $\gamma : [0,1] \to \mathbb{R}^3$ is called a streamline if $\gamma'(s) = v(\gamma(s))$ for all $s \in (0,1)$. Show that there is a function w such that

$$\frac{1}{2}|v|^2 + w(\rho)$$

stays constant along any streamline.

Hint: Use $\frac{1}{2}\nabla|v|^2 = (v \cdot \nabla)v + v \times \operatorname{curl}(v)$.

Exercise 3

Let H_1, H_2 be Hilbert spaces and $A : H_1 \to H_2$ a bounded linear operator. Show that $u_n \rightharpoonup u$ in H_1 implies $Au_n \rightharpoonup Au$ in H_2.

Exercise 4

Let H be an infinite-dimensional Hilbert space and $(u_n)_{n\in\mathbb{N}}$ an orthonormal system. Prove that $u_n \rightharpoonup 0$ but (u_n) contains no norm convergent subsequence.