Karlsruhe Institute of Technology (KIT) WS 2010/2011
Institut für Analysis
Prof. Dr. Michael Plum
Dipl.-Math.techn. Rainer Mandel

Nonlinear Boundary Value Problems: 2nd problem sheet

Exercise 5

Prove the following facts about weak convergence in a Hilbert space H:

a) $u_n \to u$ implies $u_n \rightharpoonup u$.

b) $u_n \rightharpoonup u, \|u_n\| \to \|u\|$ implies $u_n \to u$.

c) If $u_n \rightharpoonup u$ then (u_n) is bounded in H and $\|u\| \leq \liminf_{n\to\infty} \|u_n\|$.

d) (u_n) converges weakly if and only if for all $v \in H$ the sequence $\langle u_n, v\rangle$ converges.

e) Suppose $D \subset H$ is a dense subset and $\langle u_n, v\rangle \to \langle u, v\rangle$ for all $v \in D$, (u_n) bounded in H. Then $u_n \rightharpoonup u$. Give an example which shows that the result is in general not true when (u_n) is unbounded.

Hint: In c) and d) use the Banach-Steinhaus theorem.

Exercise 6

Let H be a separable real Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Characterise all continuous affine monotone mappings $A : H \to H$, i.e. all mappings A which are given by $A(u) = Tu + w$ where $T : H \to H$ is linear and continuous and $w \in H$.

Recall that a (possibly nonlinear) mapping $A : H \to H$ is called *monotone* if

$$\langle A(u) - A(v), u - v\rangle \geq 0 \qquad \forall u, v \in H.$$

Exercise 7

Prove the following statements for a montone continuous vectorfield $a : \mathbb{R}^n \to \mathbb{R}^n$:

a) If $|a(p)| \leq c(1+|p|)$ for all $p \in \mathbb{R}^n$ and some constant $c > 0$ then for every $u \in H_0^1(\Omega)$ there is a vector $w_u \in H_0^1(\Omega)$ such that

$$\langle w_u, v \rangle_{H_0^1(\Omega)} = \int_\Omega a(\nabla u) \nabla v \, dx \qquad \forall v \in H_0^1(\Omega)$$

b) Prove that the operator $A : H_0^1(\Omega) \to H_0^1(\Omega), u \mapsto w_u$ is monotone.

c) Show that $\langle y - Av, u - v \rangle \geq 0$ for all $v \in H_0^1(\Omega)$ implies $y = Au$.

d) Show that $u_n \rightharpoonup u$, $A(u_n) \rightharpoonup b$, $\langle A(u_n), u_n \rangle \to \langle b, u \rangle$ implies $Au = b$.