Karlsruhe Institute of Technology (KIT)
Institut für Analysis
Prof. Dr. Michael Plum
Dipl.-Math.techn. Rainer Mandel

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Nonlinear Boundary Value Problems: 3rd problem sheet

Exercise 8

Consider the following nonlinear boundary value problem on a bounded domain Ω

$$A(u,v) := \int_\Omega \left[a(\nabla u)\cdot\nabla v + \beta\sum_{i=1}^n b_i(u)(\partial_i u)v + c(u)v\right]dx = \int_\Omega fv\,dx \quad \forall v\in H_0^1(\Omega)$$

where $f\in L^2(\Omega)$, $\beta\in\mathbb{R}$ and

i) a is a monotone vectorfield satisfying the conditions i)-iii) from the lecture,

ii) $b_1,\dots,b_n:\mathbb{R}\to\mathbb{R}$ are bounded and continuous,

iii) $c:\mathbb{R}\to\mathbb{R}$ is continuous and the function $z\mapsto c(z)z$ is bounded from below.

Prove that for small enough $|\beta|$ the following coercivity condition holds:

$$A(u,u)\ge \delta_0\|u\|^2_{H_0^1(\Omega)} - \mu_0 \qquad \forall u\in H_0^1(\Omega)$$

for some $\delta_0>0$, $\mu_0\in\mathbb{R}$.

Exercise 9

Let the assumptions on $a, b_1,\dots,b_n, c$ from exercise 8 hold. Generalise the Galerkin method from the lecture to prove existence of a solution $u\in H_0^1(\Omega)$ of the boundary value problem

$$\int_\Omega \left[a(\nabla u)\cdot\nabla v + \beta\sum_{i=1}^n b_i(u)(\partial_i u)v + c(u)v\right]dx = \int_\Omega fv\,dx \quad \forall v\in H_0^1(\Omega)$$

given in exercise 8 provided $|\beta|$ is sufficiently small. You may argue along the followings lines:

1. Prove existence of a Galerkin approximation $u_k \in V_k = \text{span}\{w_1, \dots, w_k\}$ where $\{w_1, w_2, \dots\}$ is an orthonormal basis of $H_0^1(\Omega)$.

2. Use the estimate of exercise 8 to show that the sequence (u_k) has a weakly convergent subsequence.

3. Show that the weak limit $u \in H_0^1(\Omega)$ is a solution of the above problem. To this end use the hints given below to prove that there is a $\xi \in L^2(\Omega)^n$ and a subsequence $(v_j) := (u_{k_j})$ of (u_k) which satisfies:

   i) $\int_\Omega a(\nabla v_j) \cdot \nabla \phi \, dx \to \int_\Omega \xi \cdot \nabla \phi \, dx$ for all $\phi \in H_0^1(\Omega)$.

   ii) $\int_\Omega [\xi \cdot \nabla \phi + \beta \sum_{i=1}^n b_i(u)(\partial_i u)\phi + c(u)\phi] \, dx = \int_\Omega f \phi \, dx$ for all $\phi \in H_0^1(\Omega)$.

   iii) $\int_\Omega a(\nabla v_j) \cdot \nabla v_j \, dx \to \int_\Omega \xi \cdot \nabla u \, dx$.

   iv) $\int_\Omega (\xi - a(\nabla \phi)) \cdot (\nabla u - \nabla \phi) \, dx \geq 0$ for all $\phi \in H_0^1(\Omega)$.

Hints:

1. $H_0^1(\Omega)$ imbeds compactly into $L^2(\Omega)$. In particular if $u_k \rightharpoonup u$ then there is a subsequence (u_{k_j}) of (u_k) such that $u_{k_j} \to u$ in $L^2(\Omega)$.
2. If $g_k \to g$ in $L^p(\Omega)$ then there is a subsequence (g_{k_j}) and a function $G \in L^p(\Omega)$ such that $g_{k_j} \to g$ pointwise almost everywhere and $|g_{k_j}| \leq G$.