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Institut für Analysis
Prof. Dr. Michael Plum
Dipl.-Math.techn. Rainer Mandel

Nonlinear Boundary Value Problems: 4th problem sheet

Exercise 10

a) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Use Riesz' representation theorem to prove that for all $g \in L^2(\Omega)$ there is a unique weak solution $u \in H_0^1(\Omega)$ of

$$-\Delta u = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

which satisfies $\|u\|_{H_0^1(\Omega)} \le C_P \|g\|_{L^2(\Omega)}$ where C_P is the Poincaré constant of Ω.

b) Let $\Omega := \{(x,y) = (r\cos\phi, r\sin\phi) \in \mathbb{R}^2 : 0 < r < 1, 0 < \phi < \frac{3}{2}\pi\}$ be a domain with a reentrant corner. Show that there is a function $u \in H_0^1(\Omega)$ of the form $u(r\cos\phi, r\sin\phi) = \Theta(r^2) r^{2/3} \sin(\frac{2}{3}\phi)$ and $\Theta \in C^\infty[0,1]$ such that $u \in H_0^1(\Omega)$ is the unique weak solution of

$$-\Delta u = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

for some $g \in L^2(\Omega)$, but $u \notin H^2(\Omega)$.

Exercise 11

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

a) Assume that the function $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies

$$|f(x, z_1, p_1) - f(x, z_2, p_2)| \le L_1 |z_1 - z_2| + L_2 |p_1 - p_2|$$

for all $x \in \Omega, z_1, z_2 \in \mathbb{R}, p_1, p_2 \in \mathbb{R}^n$. Find a condition on L_1, L_2 which ensures that the boundary value problem

$$-\Delta u = f(x, u, \nabla u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

has a unique weak solution $u \in H_0^1(\Omega)$.

b) Now let $f : \mathbb{R} \to \mathbb{R}$ be continuous and assume that there is a constant $c > 0$ such that $|f(z)| \leq c(|z|+1)$ for all $z \in \mathbb{R}$. Give a sufficient condition on $\kappa > 0$ such that the boundary value problem

$$-\Delta u = \kappa f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has a weak solution $u \in H_0^1(\Omega)$.

Hint: Use exercise 10a) to rewrite the problem as a fixed point equation. Use Banach's/ Schauder's fixed point theorem in a) and b), respectively.

Exercise 12

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider the nonlinear boundary value problem

$$-\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) = f(x, u, \nabla u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{$*$}$$

where f is given as in exercise 11 and the functions $a_{ij} \in L^\infty(\Omega)$ define a symmetric matrix $A(x) := (a_{ij}(x))$ such that $\xi^T A(x)\xi \geq \lambda|\xi|^2$ for some $\lambda > 0$ and all $\xi \in \mathbb{R}^n$.

a) Write down the weak formulation for problem $(*)$.

b) Prove that the linear problem

$$-\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has a unique weak solution $u \in H_0^1(\Omega)$. Determine a constant $c > 0$ such that $\|u\|_{H_0^1(\Omega)} \leq c\|g\|_{L^2(\Omega)}$.

c) Give a sufficient condition on L_1, L_2, λ which ensures the existence of a unique weak solution of $(*)$.